From Probabilistic Mechanics to Quantum Theory - Explanation

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Abstract

The paper *"From Probabilistic Mechanics to Quantum Theory"* is written by *U. Klein*. In this paper I explain the main context that I've understood from the paper with my own words which I see is convenient. I emphasize that this file is not an "academic paper". It's just my personal notes from what I've learned and is uploaded to my personal webpage for readers to access easily. The original paper can also be found there. Please email me if there's any mistake.

1 Classical Schrödinger's Equation

It is shown that the evolution of density distribution of states in classical mechanics is given by *Liouville's equation*:

$$\frac{\partial \rho}{\partial t} - \{\mathcal{H}, \rho\} = 0$$

where $\{\}$ denotes the Poisson bracket and $\mathcal{H}$ the Hamiltonian of the system. It is obvious that if $\rho(\mathbf{r}, t)$ is a solution to this equation, then any arbitrary function of $\rho(\mathbf{r}, t)$ is also a solution to this equation (Here, $\mathbf{r}$ denotes the position vector in phase space of the system). Therefore:

$$\left(\frac{\partial}{\partial t} - \{\mathcal{H}, \cdot\} \right) f(\rho) = 0 \tag{1}$$

The classical action is given by

$$S(q, \dot{q}, t) = \int_0^{q,t} L(q, \dot{q}, t) \, \mathrm{d}t$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = L(q, \dot{q}, t)$$

$$L(q, \dot{q}, t) = \tilde{L}(q, p, t) \quad , \quad S(q, \dot{q}, t) = \tilde{S}(q, p, t)$$

$$\tilde{S}(q,p,t) = \int_0^{q,t} \tilde{L}(q,p,t)\,\mathrm{d}t$$

Hence the classical action $\tilde{S}$ in canonical co-ordinates is the solution to following equation:

$$\frac{\mathrm{d}\tilde{S}}{\mathrm{d}t} = \left(\frac{\partial}{\partial t} - \{\mathcal{H}, \cdot\}\right)\tilde{S} = \tilde{L}(q,p,t) \tag{2}$$

Since we're building up a Quantum-like theory, the preferable parameter is wave function. It is known from path integral formulation of QT that the classical wave equation is the form

$$\psi = A\ e^{\frac{i}{\hbar}\tilde{S}}$$

where S is the classical action and A is regarded as normalization constant. From definition (let's say interpretation), ψ is a function where $\|\psi\|^2$ demonstrates the probability amplitude. It suggests to work with classical parameter by the form

$$\psi \equiv \sqrt{\rho}\ e^{\frac{i}{\hbar}\tilde{S}}$$

The $\hbar$ appeared in the exponent is firstly for the dimensional flexibility of the expression and has no physical meaning yet until the quantization rule is established later.

The equation of motion for the defined "classical wave function" can be found using (1) choosing $f(\rho) = \sqrt{\rho}$ and (2):

$$\begin{aligned}
\frac{\mathrm{d}\psi}{\mathrm{d}t} &= \left(\frac{\partial}{\partial t} - \{\mathcal{H}, \cdot\}\right)\sqrt{\rho}\ e^{\frac{i}{\hbar}\tilde{S}} = \sqrt{\rho}\ e^{\frac{i}{\hbar}\tilde{S}}\ \frac{\partial\tilde{S}}{\partial t} - \sqrt{\rho}\ \{\mathcal{H}, e^{\frac{i}{\hbar}\tilde{S}}\} \\
&= \sqrt{\rho}\ e^{\frac{i}{\hbar}\tilde{S}}\ \frac{\partial\tilde{S}}{\partial t} - \frac{i}{\hbar}\sqrt{\rho}\ e^{\frac{i}{\hbar}\tilde{S}}\ \{\mathcal{H}, \tilde{S}\} \\
&= \sqrt{\rho}\ e^{\frac{i}{\hbar}\tilde{S}}\ \frac{\partial\tilde{S}}{\partial t} - \frac{i}{\hbar}\sqrt{\rho}\ e^{\frac{i}{\hbar}\tilde{S}}\ \left(\frac{\partial\tilde{S}}{\partial t} - \tilde{L}\right) = \frac{i}{\hbar}\tilde{L}\psi
\end{aligned}$$

$$-i\hbar\frac{\mathrm{d}\psi}{\mathrm{d}t} = \tilde{L}\psi$$

The actual complete equation substituting the Poisson bracket is:

$$\left[\frac{\partial}{\partial t} - \frac{\partial\mathcal{H}}{\partial q^k}\frac{\partial}{\partial p^k} + \frac{\partial\mathcal{H}}{\partial p^k}\frac{\partial}{\partial q^k} - \frac{i}{\hbar}\tilde{L}\right]\psi = 0$$

where Einstein's convention is used. Canonical Lagrangian $\tilde{L}$ is determined by Hamiltonian through

$$\tilde{L} = p^k\frac{\partial\mathcal{H}}{\partial p^k} - \mathcal{H}$$

Substituting the Lagrangian:

$$\left[\frac{\partial}{\partial t} - \frac{\partial\mathcal{H}}{\partial q^k}\frac{\partial}{\partial p^k} + \frac{\partial\mathcal{H}}{\partial p^k}\frac{\partial}{\partial q^k} - \frac{i}{\hbar}\left(p^k\frac{\partial\mathcal{H}}{\partial p^k} - \mathcal{H}\right)\right]\psi = 0$$

where is rearranged to

$$\left[\frac{\partial}{\partial t} - \frac{\partial \mathcal{H}}{\partial q^k}\frac{\partial}{\partial p^k} + \frac{\partial \mathcal{H}}{\partial p^k}\left(\frac{\partial}{\partial q^k} - \frac{i}{\hbar}p^k\right) + \frac{i}{\hbar}\mathcal{H}\right]\psi = 0 \tag{3}$$

The quantization rule is to "project" the phase space into configuration space. Hence equation of motion contains no p^k. For this to happen, the following conditions must be satisfied:

$$\boxed{\frac{\partial \psi}{\partial p^k} = 0 \quad , \quad -i\hbar\frac{\partial}{\partial q^k} \equiv p^k} \tag{4}$$

and the Quantum-Mechanical counterpart of (3) according to quantization rules (4) is given by Schrödinger's Equation for Hamiltonian $\mathcal{H}(q,p)$ by:

$$\boxed{-i\hbar\frac{\partial \psi}{\partial t} = \mathcal{H}\left(q, -i\hbar\frac{\partial}{\partial q^k}\right)\psi} \tag{5}$$

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