Symmetric : $\langle u|Av\rangle = \langle Au|v\rangle \ \forall\, v,u \in D_A$; $\mathcal{H} = \mathcal{L}^2([0,2\pi])$

$u(0) \neq u(2\pi)$

$P = -i\, d/dx$; $P^* = -i\, d/dx$ $D_{P^*} \subset D_P$

$P^* \to$ symmetric $P \to$ not symmetric

expanding the domain : Sym $\longrightarrow$ self-adjoint

$\mathcal{H} = \mathcal{L}^2([0,2\pi])$

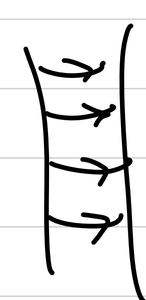
$A_\alpha = -i\, d/dx$ D_α : $\underline{v(2\pi) = e^{2\pi i \alpha}\, v(0)}$; $0 \leq \alpha < 1$

$\underline{A_\alpha^* = A_\alpha}$: $u(2\pi)^*\, v(2\pi) - u(0)^*\, v(0) = 0$

$A_\alpha^* \longrightarrow$ self-adjoint

spectra: $-i\, d/dx\, v = \lambda v \Rightarrow \underline{v(x) = e^{i(m+\alpha)x}}$; $m \in \mathbb{Z}$

$\underline{\lambda = m + \alpha}$



Aharonov – Bohm:

$\vec{B} = \nabla \times \vec{A} \Rightarrow \Phi = \int \vec{B}\cdot d\vec{S} = \int \nabla\times\vec{A}\cdot d\vec{S} = \oint \vec{A}\cdot d\vec{l}$

$\vec{A} = \dfrac{\Phi}{2\pi r}\,\hat{\theta}$

$H = \dfrac{1}{2m}\left(\vec{P} - \dfrac{q}{c}\vec{A}\right)^2 = \dfrac{1}{2m}\left(P_r^2 + \dfrac{1}{r^2}\left(P_\theta - \underline{\dfrac{q}{c}\dfrac{\Phi}{2\pi}}\right)^2 + P_z^2\right)$

Classical Mech. : $P_\theta \longrightarrow P_\theta + \frac{q}{c}\frac{\Phi}{2\pi} \implies H = \frac{P^2}{2m}$

QT : $P_\theta = -i\hbar \,\partial/\partial\theta$. $\psi = e^{\frac{i}{\hbar}\frac{q\Phi}{2\pi c}\theta}\chi$

$$\left(-i\hbar\,\partial/\partial\theta - \frac{q}{c}\frac{\Phi}{2\pi}\right)\psi = e^{\frac{i}{\hbar}\frac{q\Phi}{2\pi c}\theta}\left(-i\hbar\,\partial/\partial\theta\right)\chi$$

$H\psi = E\psi \implies \frac{p^2}{2m}\chi = E\chi$, $\underline{\chi(\theta=2\pi) = e^{-\frac{i}{\hbar}\frac{q\Phi}{c}}\chi(0)}$

$\psi(\theta=2\pi) = \psi(\theta=0)$

Radial Momentum: $P_r = -i\, d/dr$; $\mathcal{H} = \mathcal{L}^2([0,\infty])$

D_{P_r} = all differentiable function in $0 \leq r < \infty$: $\langle u|v\rangle = \int_0^\infty u^* v \, dr$

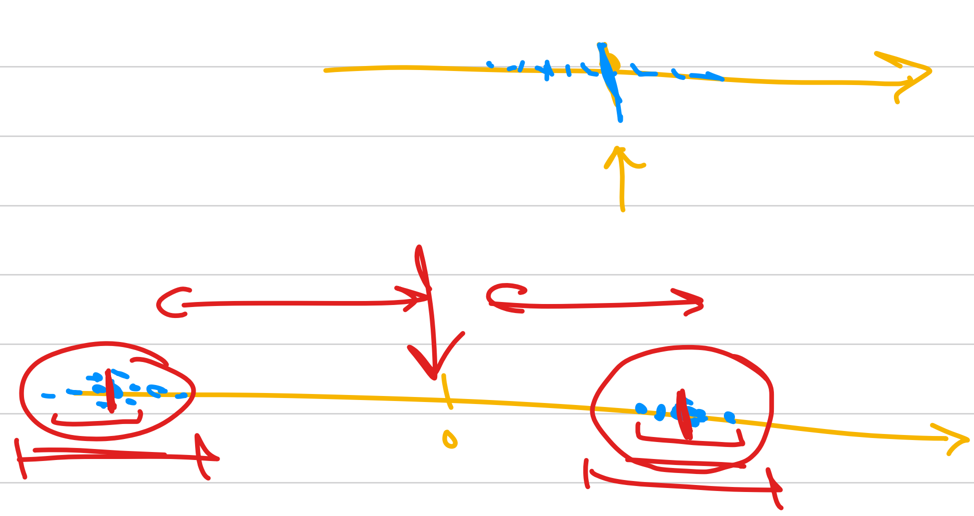
$u(\infty) = 0$

$\underline{\underline{u(0) = 0}}$ → to be symmetric

$\hat{P}_r = -i\, d/dr$, D_{P_r} : $v(0) = 0 \implies \hat{P}_r^*$ need not to satisfy $v(0) = 0$!

$D_{P_r} \subset D_{P_r^*}$

uncertainty relations :



$$\Delta A \, \Delta B = \frac{1}{2} \left| \langle AB - BA \rangle \right|$$

Proof :

$$\|Au\|^2 \, \|Bu\|^2 \geq |\langle Au | Bu \rangle|^2$$

$$\langle Au|Bu\rangle = \langle u | A^* B u \rangle = \underbrace{\langle u | \frac{A^*B + B^*A}{2} | u \rangle}_{\text{Real}} + \underbrace{\langle u | \frac{A^*B - B^*A}{2} | u \rangle}_{\text{Imag.}}$$

$A^*B + B^*A \longrightarrow$ sym $\Longrightarrow$

$i(A^*B - B^*A) \longrightarrow$

$$|\langle Au|Bu\rangle|^2 = \frac{1}{4} |\langle A^*B + B^*A\rangle|^2 + \frac{1}{4} |\langle A^*B - B^*A\rangle|^2$$

$$\|Au\|^2 = \langle Au | Au \rangle = \langle u | A^*A | u \rangle = \langle A^*A \rangle$$

$$\langle A^*A\rangle^2 \langle B^*B\rangle^2 \geq \frac{1}{4} |\langle A^*B + B^*A\rangle|^2 + \frac{1}{4} |\langle A^*B - B^*A\rangle|^2 \geq \frac{1}{4} |\langle A^*B - B^*A\rangle|^2$$

$$A \to A - \underbrace{\langle A \rangle}_{a} =: \tilde{A}$$

$$B \to B - \langle A \rangle =: \tilde{B}$$

$$\langle \tilde{A}^* \tilde{A} \rangle = \langle \underbrace{(A-a)^2}_{-2aA} \rangle = \langle A^2 - 2aA + a^2 \rangle$$

$$[\tilde{A}, \tilde{B}] = [A, B]$$

$$= \langle A^2 \rangle - 2a^2 + a^2 = \langle A^2 \rangle - \langle A \rangle^2 = \Delta A$$

$$\langle \tilde{A}^* \tilde{A} \rangle = \Delta A \longrightarrow (\Delta A)^2 (\Delta B)^2 \geq \frac{1}{4} \left| \langle A^* B - B^* A \rangle \right|^2$$

$$\boxed{(\Delta A)(\Delta B) \geq \frac{1}{2} \left| \langle AB - BA \rangle \right|}$$

$u \in D_A$, $u \in D_B$

$u \in D_{A^*}$, $u \in D_{B^*}$

$$(A - \langle A \rangle) u \parallel (B - \langle B \rangle) u$$

$$\langle \tilde{A}^* \tilde{B} + \tilde{B}^* \tilde{A} \rangle = 0 \longrightarrow \langle u | \tilde{A}^* \tilde{B} | u \rangle + \langle u | \tilde{B}^* \tilde{A} | u \rangle = 0$$

$\hookrightarrow \text{Real}\{ \langle A^* B \rangle \} = 0$

$$\frac{\alpha}{\beta} \underbrace{(A - \langle A \rangle) u}_{v} + \underbrace{(B - \langle B \rangle) u}_{w} = 0$$

$\text{Real}\{ \langle v | w \rangle \} = 0$

$\langle A^* B \rangle = \langle u | A^* B | u \rangle$

$= \langle \underbrace{Au}_{v} | \underbrace{Bu}_{w} \rangle$

$$\frac{\alpha}{\beta} v + w = 0 \longrightarrow w = -\frac{\alpha}{\beta} v \Longrightarrow$$

$$1 = -\frac{\alpha}{\beta} \langle w | v \rangle \longrightarrow \frac{\alpha}{\beta} : \text{Pure imaginary} \longrightarrow i\lambda$$

P & X : $\Delta P \, \Delta X \geq \frac{\hbar}{2}$

$$\underbrace{\phi(x) = C e^{-ax^2 - bx}}$$ minimum uncertainty

$$\hat{a} | a \rangle = \alpha \underbrace{| a \rangle}$$ Coherent states