$$[X, P] = i\hbar \hat{\mathbb{1}} \Rightarrow \mathrm{Tr}[XP - PX] = \mathrm{Tr}[XP] - \mathrm{Tr}[PX] = 0$$

$$\mathrm{Tr}[\hat{\mathbb{1}}] = d \Rightarrow d = 0 \quad !$$

Linear operators:

$$A_{mn} := \delta_{1,m} + \delta_{1,n} \qquad A = \begin{pmatrix} 2 & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & & \\ 1 & 0 & \ddots & & \\ \vdots & & & & \end{pmatrix}$$

$$|v\rangle = A|u\rangle \rightarrow v_i = \sum_j A_{ij} u_j$$

$$v_1 = 2u_1 + \overbrace{\sum_i u_i}^{\text{may diverge!}} \qquad \text{even if } \sum_i |u_i|^2 < \infty \qquad \text{ex. } u_n = \frac{1}{n}$$

$$m \neq 1 \qquad v_m = u_1$$

$$\|v\| = \sum_n |v_n|^2 = \Big|2u_1 + \underbrace{\sum_i u_i}\Big|^2 + |u_1|^2 \overbrace{\sum_{n=2}^{\infty} 1}$$

if A is defined $\Rightarrow$ A^2 doesn't exist !

Domain of definition.

$$A \rightarrow D_A . \qquad D_A := \{ |u\rangle \in \mathcal{H} \mid A|v\rangle \in \mathcal{H} \}$$

Theorem:

$$D_A = \mathcal{H} \Longleftrightarrow \|A|u\rangle\| \text{ is bounded } \forall |u\rangle : \|u\| = 1$$

+ closed

To be Proved ...

$$\|A\| := \sup\left\{ \frac{\|Au\|}{\|u\|} \right\} \qquad \forall\, |u\rangle \neq 0$$

$$D_A = \mathcal{H} \iff \|A\| < \infty \qquad : \text{ Bounded}$$

local & quasi-local operators :

$$|v\rangle = A|u\rangle \rightarrow v_i = \sum_j A_{ij} v_j$$

$$v(x) = \int_{-\infty}^{\infty} A(x,y)\, u(y)\, dy$$

$$A_{mn} = a_m \delta_{mn} \Rightarrow A(x,y) = a(x)\,\delta(x-y) \Rightarrow \text{local}$$

$$\Rightarrow v(x) = \int A(x,y)\, u(y)\, dy = a(x)\, u(x)$$

$$v_i = a_i u_i$$

$$B = \begin{pmatrix} 0 & x & & & \\ -x & 0 & x & & \\ & -x & 0 & x & \\ & & -x & 0 & \\ & & & & \ddots \end{pmatrix} \rightarrow B_{mn} = b_m\left(\delta_{m,n+1} - \delta_{m,n-1}\right)$$

$$\partial/\partial y$$

$$B(x,y) = b(x)\,\delta'(x-y) \rightarrow v(x) = \int b(x)\,\delta'(x-y)\, u(y)\, dy$$

$$v(x) = \int b(x)\, \delta(x-y)\, u'(y)\, dy = b(x)\, u'(x)$$

$$B \to \underline{b(x)\frac{d}{dx}} \to X \text{ rep. of } B$$

↳ unbounded !

Counter example : $\mathcal{H} = \mathcal{L}^2([0,2\pi])$ $\langle f|g\rangle = \int_0^{2\pi} f^*(x) g(x)\, dx$

$$W_m(x) = \frac{e^{imx}}{\sqrt{2\pi}} \to \langle W_m | W_m\rangle = 1$$

$$\underline{A = -i\, d/dx} \to A\, W_m(x) = m\, W_m(x)$$

$$\| A W_m \| = m^2 \to \text{unbounded !}$$

$m \to \infty$: $\|A W_m\| \to \infty$

$$\psi(x) = e^{\frac{-ix^2}{2}} \frac{\sin x}{x} \notin D_{\hat{p}} \qquad H = \frac{P^2}{2M}$$

$$K(x,x';t,t_0) = \langle x | \hat{U}(t,t_0) | x'\rangle \qquad U = e^{-\frac{i}{\hbar} t H}$$

$$U^+ = U^{-1} \Rightarrow \|U\| = 1 < \infty$$

$D_{\hat{A}+\hat{B}} = D_{\hat{A}} \cap D_{\hat{B}}$; $D_{c\hat{A}} = D_{\hat{A}}$; $D_{\hat{A}\hat{B}} = (B \text{ is defined}) \in D_{\hat{A}}$

Adjoint: $\hat{A} \to \hat{A}^*$ $\qquad \underline{\langle u | Av \rangle = \langle A^* u | v \rangle} \quad \forall u, v \in \mathcal{H}$

$$\left.\begin{array}{l} A_1^* : \langle A_1^* u | v \rangle = \langle u | Av \rangle \\ A_2^* : \langle A_2^* u | v \rangle = \langle u | Av \rangle \end{array}\right\} \quad \langle (A_1^* - A_2^*) u | v \rangle = 0 \quad \forall u, v \in \mathcal{H}$$

$\hookrightarrow A_1^* - A_2^* = 0$

$\hookrightarrow A_1^* = A_2^*$

$\langle u | Bv \rangle = 0 \quad \forall u, v \in \mathcal{H} \iff B = 0$

$|n\rangle$: basis

$$\underbrace{\sum_n |n\rangle\langle n|}_{\mathbb{1}} Bv\rangle = 0 \implies B|v\rangle = 0 \quad \forall |v\rangle$$

$\hookrightarrow B = 0$

Hermitian
- Self-adjoint : $A^* = A \iff D_A = D_{A^*}$
- Symmetric : $\langle Au | v \rangle = \langle u | Av \rangle$

$$\langle f | g \rangle = \int_0^{2\pi} f^* g \, dx$$

ex.

$A = -i \, d/dx \qquad D_A$: Differentiable functions $\qquad \mathcal{H} = \mathcal{L}^2([0, 2\pi])$

$A^* = -i \, d/dx \longrightarrow D_{A^*}$. Dif. fun. where $\underline{u(0) = u(2\pi) = 0}$?

$$D_{A^*} \subset D_A$$

$$\langle u | Av \rangle - \langle A^* u | v \rangle = -i \int_0^{2\pi} \underbrace{(u^* v' - u^{*\prime} v)}_{\frac{d}{dx}(u^* v)} \, dx = -i \left(u^*(2\pi) v(2\pi) - u^*(0) v(0) \right) \overset{?}{=} 0$$

$$\underline{u(2\pi) = u(0) = 0} \longleftrightarrow D_A = D_{A^*} \longleftarrow A^* = A$$

A is an extension of A^*

A^* is a restriction of A

A is closed iff : whenever $\psi_n \longrightarrow \psi$ & $A\psi_n \longrightarrow \phi$

then : $\psi \in D_A$ & $A\psi = \phi$

$$D_A = \mathcal{H} \Leftrightarrow \underbrace{\sup\{\|A\psi\|\} < \infty \quad \forall |\psi\rangle \in \mathcal{H} \mid \|\psi\| = 1}_{\|A\| < \infty}$$

$+ A$ is closed

$\Leftarrow$: $\forall |\psi\rangle \in \mathcal{H} \mid \|\psi\| = 1 : \exists A|\psi\rangle \in \mathcal{H}$

$$|\phi\rangle = \|\phi\| \frac{|\phi\rangle}{\|\phi\|} \Longrightarrow A|\phi\rangle = \|\phi\| \overbrace{A \frac{|\phi\rangle}{\|\phi\|}}^{\in \mathcal{H}} \in \mathcal{H}$$

$\Rightarrow$:

$D_A = \mathcal{H}$; Assume $\|A\psi\|$ is unbounded

$$\forall n \in \mathbb{N} . \exists |\psi_n\rangle \in \mathcal{H} : \|A\psi_n\| > n$$

$$|\chi_n\rangle := \frac{|\psi_n\rangle}{\|A\psi_n\|} \Longrightarrow \|\chi_n\| = \frac{1}{\|A\psi_n\|} < \frac{1}{n}$$

$$0 \leqslant \|\chi_n\| < \frac{1}{n} \longrightarrow \underline{\lim_{n\to\infty} \|\chi_n\| = 0 \ : \ \chi_n \to 0}$$

$$A|\chi_n\rangle = \frac{A|\psi_n\rangle}{\|A\psi_n\|} = 1 \Longrightarrow \underline{\lim_{n\to\infty} \|A\chi_n\| = 1} \neq \|A\,0\| \cdot \times$$

even if A is closed $\longrightarrow$ A^* is closed

$$\psi_n \to \psi : \quad \forall|v\rangle : \langle v|A^*|\psi_n\rangle = \langle Av|\psi_n\rangle$$

$$\langle Av|\psi_n\rangle \longrightarrow \langle Av|\psi\rangle = \langle v|A^*\psi\rangle$$

$$\forall|v\rangle : \langle v|A^*\psi_n\rangle \longrightarrow \langle v|A^*\psi\rangle$$

$$\sum_n |n\rangle\langle n|A^*\psi_n\rangle \to \sum_n |n\rangle\langle n|A^*\psi\rangle \Longrightarrow \underline{A^*|\psi_n\rangle \longrightarrow A^*|\psi\rangle} \ \checkmark$$

Can be Proved : $\begin{cases} \text{if } A \text{ is closed } \& \ D_A : \text{dense in } \mathcal{H} \Longrightarrow D_{A^*} \text{ is dense} \\ A^{**} = A \end{cases}$