$$P_i \longrightarrow |\psi_i\rangle \qquad\qquad \langle\hat{A}\rangle_i = \langle\psi_i|A|\psi_i\rangle$$

$$\langle A\rangle = \sum_i P_i \langle A\rangle_i = \sum_i P_i \langle\psi_i|A|\psi_i\rangle = \sum_n \sum_i P_i \langle\psi_i|n\rangle\langle n|A|\psi_i\rangle$$

$$= \sum_n \langle n|\left(\sum_i P_i A|\psi_i\rangle\langle\psi_i|\right)|n\rangle$$

$$= \mathrm{Tr}\Big[A\underbrace{\sum_i P_i |\psi_i\rangle\langle\psi_i|}_{\hat{\rho}}\Big] = \mathrm{Tr}[\hat{A}\hat{\rho}] = \underbrace{\mathrm{Tr}[\hat{\rho}\hat{A}] = \langle\hat{A}\rangle}$$

$$\rho := \sum_i P_i |\psi_i\rangle\langle\psi_i| \longrightarrow \underbrace{\mathrm{Tr}[\hat{\rho}] = \langle\hat{1}\rangle}$$

Pure : $\underbrace{\hat{\rho} = |\phi\rangle\langle\phi|}$

$$\hat{\rho}^2 = \hat{\rho}$$

$$|\phi\rangle\underbrace{\langle\phi|\phi\rangle}_{1}\langle\phi| = |\phi\rangle\langle\phi|$$

$\underline{\underline{\hat{\rho}^2 = \hat{\rho}}}$: $\exists\, |\phi\rangle$. $\hat{\rho} = |\phi\rangle\langle\phi|$

$$\lambda \in \{0,1\} \quad , \quad \rho = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \; 0 \; 1 \; 1 \ldots \end{pmatrix} \Longrightarrow \mathrm{Tr}[\hat{\rho}] = 1$$

$$\exists\, |w\rangle \; : \; \hat{\rho}|w\rangle = |w\rangle$$

$$\Rightarrow \underline{\hat{\rho} = |w\rangle\langle w|}$$

$$\langle v|\hat{\rho}|v\rangle = \langle v|\hat{\rho}^2|v\rangle = \|\hat{\rho}|v\rangle\|^2 \geq 0 \qquad \forall\, |v\rangle$$

$$\hat{\rho} = \sum_i P_i |u_i\rangle\langle u_i| \longrightarrow \langle v|\hat{\rho}|v\rangle = \sum_i P_i \langle v|u_i\rangle\langle u_i|v\rangle = \sum_i P_i |\langle v|u_i\rangle|^2 \geq 0$$

$$\forall |w\rangle : \quad \langle w|A|w\rangle \in \mathbb{R} \quad \therefore \quad A = A^{+}$$

$$|v\rangle = \sum_n v_n \left(\delta_{n,\alpha} + \delta_{n,\beta}\right)|n\rangle = v_\alpha|\alpha\rangle + v_\beta|\beta\rangle$$

$$\forall |v\rangle \in \mathcal{H} : \quad \langle v|A|v\rangle \geq 0 \implies \hat{A} \geq 0$$

$$\langle v|A|v\rangle = \begin{pmatrix} v_\alpha^* & v_\beta^* \end{pmatrix} \underbrace{\begin{pmatrix} A_{\alpha\alpha} & A_{\alpha\beta} \\ A_{\beta\alpha} & A_{\beta\beta} \end{pmatrix}}_{\hat{A}' \geq 0} \begin{pmatrix} v_\alpha \\ v_\beta \end{pmatrix} \geq 0$$

$$\det \hat{A}' = \prod_i \lambda_i \qquad \lambda_i \geq 0 \; \Bigg\} \; \det A' \geq 0 \rightarrow A_{\alpha\alpha} A_{\beta\beta} - |A_{\alpha\beta}|^2 \geq 0$$

$$A_{\alpha\alpha} = 0 \longrightarrow A_{\alpha\beta} = 0 \quad \forall \beta$$

Pure. $\quad \rho = \lambda \rho' + (1-\lambda)\rho'' = |u\rangle\langle u|$

$$|v\rangle : \quad \langle v|u\rangle = 0 \implies \underset{\langle v|u\rangle\langle u|v\rangle}{\langle v|\hat{\rho}|v\rangle} = \underset{\geq 0}{\lambda} \underbrace{\langle v|\hat{\rho}'|v\rangle}_{\geq 0} + \underbrace{(1-\lambda)}_{\geq 0} \underbrace{\langle v|\rho''|v\rangle}_{\geq 0} = 0$$

$$\implies \rho = \rho' = \rho''$$

$$\begin{pmatrix} 0.7 & 0 \\ 0 & 0.3 \end{pmatrix} = 0.7 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\binom{1}{0}} + 0.3 \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\binom{0}{1}}$$

$$= 0.4 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0.3 \begin{pmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{pmatrix} + 0.3 \begin{pmatrix} 0.36 & -0.48 \\ -0.48 & 0.64 \end{pmatrix}$$

$$|\pm\rangle = \frac{|x\rangle \pm i|y\rangle}{\sqrt{2}}$$

$$\begin{cases} 50\% \longrightarrow |x\rangle \\ 50\% \longrightarrow |y\rangle \end{cases} \Longrightarrow \hat{\rho} = \sum_i p_i |u_i\rangle\langle u_i| = \frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}$$

$$= \frac{1}{2}\hat{\mathbb{1}}$$

$$\left.\begin{cases} 50\% \longrightarrow |+\rangle \\ 50\% \longrightarrow |-\rangle \end{cases}\right\} \hat{\rho} = \frac{1}{2}\hat{\mathbb{1}}$$

$$\underline{\underline{\rho_1}}, \quad \underline{\underline{\rho_2}} \qquad \begin{cases} \lambda \longrightarrow \rho_1 \\ 1-\lambda \longrightarrow \rho_2 \end{cases}$$

$$\underbrace{\hat{\rho} = \lambda \rho_1 + (1-\lambda)\rho_2} = \lambda \sum_i p_i^{(1)} |u_i^{(1)}\rangle\langle u_i^{(1)}| + \cdots$$

$$\lambda p_i^{(1)} \longrightarrow |u_i^{(1)}\rangle$$

Determination of Density matrix

2D system:

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ; \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \lambda = \pm 1$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}, \lambda = \pm 1$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}i\\1\end{pmatrix}, \lambda = \pm 1$$

$$a_j = \langle \sigma_j \rangle = \mathrm{Tr}[\rho\, \sigma_j] \longrightarrow 3+1 \text{ eq.} \qquad \mathrm{Tr}[\hat{\rho}] = 1$$

$$\hat{\rho} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longrightarrow \alpha, \beta, \beta, \beta^* \Rightarrow 4$$

$\gamma = \beta^*$

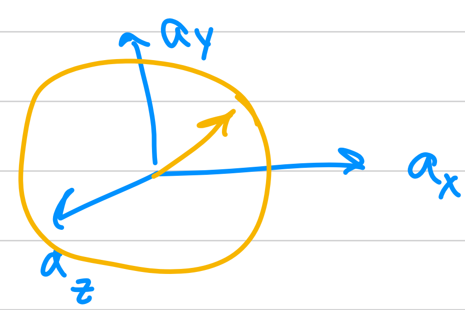
$$\boxed{\hat{\rho} = \frac{1}{2}\left(\hat{1} + \sum_j a_j \hat{\sigma}_j\right)} \Rightarrow a_i = Tr[\hat{\rho}\,\sigma_i] = \frac{1}{2}\sum_j a_j \, Tr[\sigma_j \sigma_i]$$

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\hat{1}$$

$$Tr[AB] = Tr[BA] \Rightarrow Tr[\sigma_i\sigma_j] = Tr\left[\frac{\overbrace{\sigma_i\sigma_j + \sigma_j\sigma_i}^{2\delta_{ij}\hat{1}}}{2}\right] = Tr[\hat{1}]\,\delta_{ij} = 2\delta_{ij}$$

$$a_i = \sum_j a_j \delta_{ij} = a_i \;\checkmark$$

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \; |\vec{a}| \leq 1$$



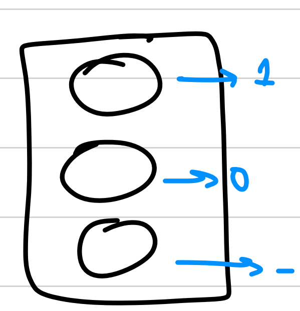
$2+3+4+\cdots+N-1$

$$\rho = \begin{pmatrix} * & * & * & - - \\ * & * & & \\ * & * & \ddots & \\ \vdots & & & \end{pmatrix}_{N\times N} \longrightarrow N + \frac{N(N-1)}{2}\times 2 = N^2$$

$$Tr[\hat{\rho}] = 1$$

$$\underline{\underline{N^2-1}} \to \langle \sigma_j \rangle \quad ; \quad \hat{A} = \sum_k a_k |a_k\rangle\langle a_k|$$

$$\hat{\rho}: \quad \underline{\underline{P_k}} = Tr[\hat{\rho}\,|a_k\rangle\langle a_k|] \longrightarrow \underline{\underline{N-1 \text{ eq}}}$$



$$\sum P_k = 1$$

$$\frac{N^2-1}{N-1} = N+1 \Rightarrow \text{measurements}$$

Continuous Variables

$$\langle u|v\rangle = \int_{-\infty}^{\infty} u^*(x)\, v(x)\, dx \longrightarrow \langle v|v\rangle \geq 0$$

$\int |v|^2 dx \longrightarrow$ Converge

$\int |v|^2 dx < \infty \implies$ square integrable

e^{kx} $\quad$ e^{-kx}

$$\psi(x) = e^{ikx} \longrightarrow \psi \stackrel{?}{=} e^{kx}$$

$$\psi(x) = \int g(k)\, e^{ikx}\, dk \quad ; \quad E \geq \min(V)$$

$$4\langle u|v\rangle = \|u+v\|^2 - \|u-v\|^2 + i\|u-iv\|^2 - i\|u+iv\|^2$$

Strong Convergence : $\{|u_n\rangle\}$

if $\|u_m - u_n\| \to 0$, $m,n\to\infty \implies \exists u \in \mathcal{H} : \|u_m - u\| \to 0$

Weak Convergence : $\forall |v\rangle$: $\langle v|u_n\rangle \longrightarrow \langle v|u\rangle$

square - summable function : $|v\rangle = \sum_{i=1}^{\infty} v_i |e_i\rangle$

$\sum_i |v_i|^2 < \infty \implies v_{i\to\infty} = 0$

$\{|e_n\rangle\}$: $\forall |v\rangle$: $\underbrace{\langle v|e_i\rangle}_{\to 0}$ $\quad i\to\infty$

$|e_i\rangle \neq 0$

$\sqrt{\delta(x)}$: $u_n(x) = \begin{cases} \sqrt{n} & |x| \leq \frac{1}{2n} \\ 0 & |x| \geq \frac{1}{2n} \end{cases}$

Weakly $\to 0$

$i \frac{\partial}{\partial t} \psi = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \psi$; $\hbar \equiv 1$ $m=1$

$|\psi(t)\rangle = \hat{U}(t, t_0) |\psi(t_0)\rangle$: $U(t, t_0) = e^{-\frac{i}{\hbar} t \hat{H}}$

$$\psi(x,t) = \int \underbrace{K(x, x'; t)}_{\text{Propagator}} \psi(x', 0) \, dx'$$

$$K(x, x', t) = \frac{1}{\sqrt{2\pi i t}} e^{\frac{i(x-x')^2}{2t}}$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi i t}} \int_{-\infty}^{\infty} e^{\frac{i(x-x')^2}{2t}} \psi(x', 0) \, dx'$$

$$\psi(x, 0) = \frac{e^{-\frac{i}{2}x^2}}{(1+x^2)^{1/3}}$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi i t}} \int e^{\frac{i}{2t}(x^2 + x'^2 - 2xx')} \frac{e^{-i\frac{x'^2}{2}}}{(1+x'^2)^{1/3}} \, dx'$$

$\psi(0,1) \to \infty$: $\psi(x,1) = \frac{2\sqrt{\pi}}{\Gamma(1/3)} \left(\frac{|x|}{2}\right)^{1/6} K_{1/6}(x)$