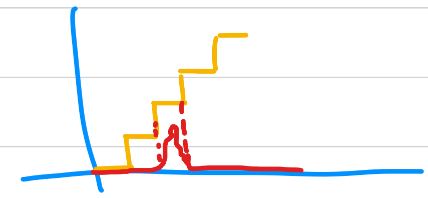
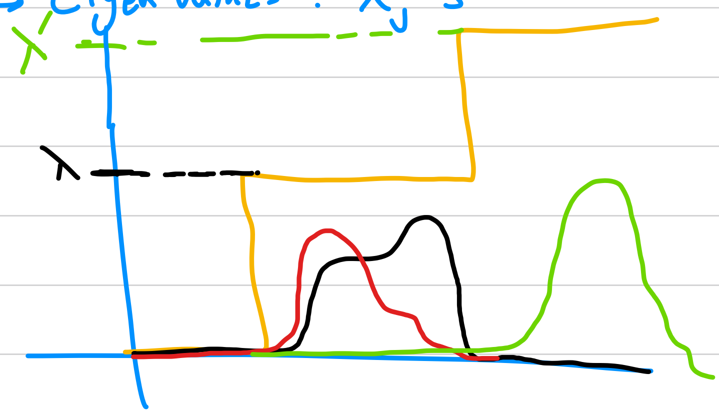
Spectral Theory :

x : Continuous Variable



$$x' = f(x) = x_j \quad \forall \; x_j \leq x \leq x_{j+1}$$

Operator $\hat{X}$: multiplication by $f(x)$ $\longrightarrow$ Eigen Values : x_j's



Quantum Test : Y/N ($Y \to 1$, $N \to 0$) : is $x_j \leq X < x_{j+1}$?

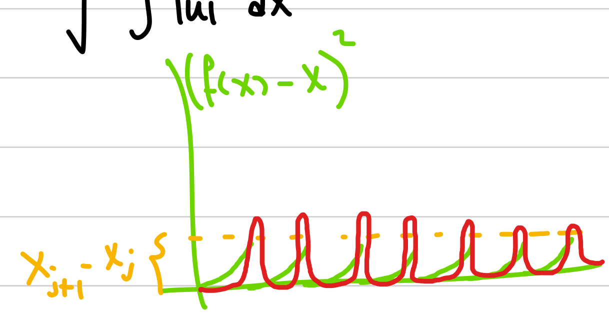
$$\hat{P}_j(x) = \begin{cases} 1 & \text{if} \quad x_j \leq x < x_{j+1} \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

$$\hat{P}_j \hat{P}_k = \delta_{jk} \hat{P}_k \quad ; \quad \sum_j \hat{P}_j = \hat{\mathbb{1}}$$

$$\hat{X} = \sum_j x_j \hat{P}_j \quad ; \quad \|A\| = \sup \left\{ \frac{\|Au\|}{\|u\|} \right\}$$

$$\|\hat{X} - x\hat{\mathbb{1}}\| = \sup \left\{ \frac{\|(\hat{X} - x)u\|}{\|u\|} \right\} = \sup \left\{ \frac{\sqrt{\int (f(x) - x)^2 |u|^2 dx}}{\sqrt{\int |u|^2 dx}} \right\}$$

$$= |x_{j+1} - x_j|$$



Spectral family :

$$\hat{E}(x_j) = \sum_{k=0}^{j-1} \hat{P}_k \quad : \text{ is } X < x_j \text{ ?} \quad \overset{\text{Yes}}{\rightarrow} 1, \quad \underset{\text{no}}{\rightarrow} 0$$

$$\hat{E}(x_{j+1}) = \hat{E}(x_j) + \hat{P}_j \quad ; \quad \hat{E}(x_{max}) = \hat{\mathbb{1}}$$

$$E(x_{min}) = 0$$

$$\hat{E}(x_n)\,\hat{E}(x_m) = \hat{E}(x_m)\,\hat{E}(x_n) = \begin{cases} \hat{E}(x_m) & x_m \leq x_n \\ \hat{E}(x_n) & x_m \geq x_n \end{cases}$$

Continuum limit :

$$\hat{E}(\xi) \quad . \quad \text{is } \xi < X \text{ ?}$$

$$d\hat{E}(\xi) := \hat{E}(\xi + d\xi) - \hat{E}(\xi) \quad ; \quad d\hat{E}(\xi) : \text{ is } \xi \leq X < \xi + d\xi$$

$$\hat{X} = \sum_j X_j P_j \implies \hat{X} = \int_0^{\hat{1}} \xi \, d\hat{E}(\xi) \rightarrow \text{spectral } \frac{\text{decomposition}}{\text{resolution}}$$

Stieltjes integrals

$\hat{E}(\xi)$. $\frac{\text{spectral family}}{\text{resolution of identity}}$ generated by X

$$f(\hat{X}) := \int_0^{\hat{1}} f(\xi)\, d\hat{E}(\xi) = \int_{x_{min}}^{x_{max}} f(\xi) \frac{d\hat{E}(\xi)}{d\xi} d\xi$$

* if f is real : $f(\hat{X})$ is self-adjoint

$$\langle u | f(\hat{X}) v \rangle = \langle f(\hat{X}) u | v \rangle \; ; \quad \langle u | \hat{E}(\xi) v \rangle = \int_{-\infty}^{\infty} u^*(x)\, \hat{E}(\xi)\, v(x)\, dx$$

$$\langle u | \hat{E}(\xi) v \rangle = \int_{\xi}^{\infty} u^*(x)\, v(x)\, dx$$

$$\langle \hat{E}(\xi) u | v \rangle = \nearrow \qquad \square$$

$$\left(\int f(\xi)\, d\hat{E}(\xi)\right)\left(\int g(\eta)\, d\hat{E}(\eta)\right) = \int f(\sigma) g(\sigma)\, d\hat{E}(\sigma)$$

$$\int f(\xi) g(\eta)\, \underbrace{d\hat{E}(\xi)\, d\hat{E}(\eta)} =$$

$d\hat{E}(\eta) \quad \eta < \xi$

$dE(\xi) \quad \eta > \xi \qquad \square$

$$A = \int f(\xi)\, d\hat{E}_A(\xi)$$

$$B = \int g(\eta)\, d\hat{E}_B(\eta)$$

if $E_A = E_B \Longrightarrow \exists h : A = h(B)$

$$A = f \circ g^{-1}(B) \quad \square$$

Classification of spectra:

A : Continous spectra,

$$\| (A - \lambda) \psi_\lambda \| < \varepsilon$$

$$P_\lambda = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} d\hat{E}(\xi) \longrightarrow \underline{P_\lambda \psi_\lambda = \psi_\lambda}$$

$$P_\lambda \phi \neq 0 \to \psi_\lambda = \frac{P_\lambda \phi}{\|P_\lambda \phi\|}$$

$$A\psi_\lambda = AP_\lambda \psi_\lambda = \left(\int \xi \, dE(\xi)\right)\left(\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} dE(\eta)\right)\psi_\lambda$$

$$= \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \xi \, dE(\xi) \, \psi_\lambda \qquad \hat{1} = \int dE(\xi)$$

$$\|(A-\lambda)\psi_\lambda\|^2 = \left\| \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (\xi-\lambda)\, dE(\xi)\, \psi_\lambda \right\|^2$$

$$= \langle \psi_\lambda | \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (\xi-\lambda)^2 \, dE(\xi) | \psi_\lambda \rangle$$

$$= \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (\xi-\lambda)^2 \langle \psi_\lambda | dE(\xi) | \psi_\lambda \rangle \le \varepsilon^2 \langle \psi_\lambda | \underbrace{\int dE(\xi)}_{\hat{1}} | \psi_\lambda \rangle$$

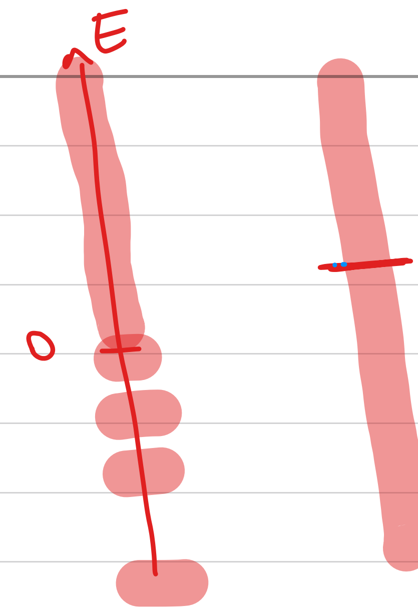
$$\boxed{\|(A-\lambda)\psi_\lambda\| \le \varepsilon}$$

$$\|(f(\hat{A}) - f(\lambda))\psi_\lambda\| = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (f(\xi) - f(\lambda))\, d\hat{E}(\xi)\, \psi_\lambda \|$$

$$\|(f(\hat{A}) - f(\lambda))\psi_\lambda\| \le \varepsilon |f'(\lambda)| + O(\varepsilon^2)$$

$$\langle f(\hat{A}) \rangle \longrightarrow f(\lambda)$$

$$f(\hat{A})\psi_\lambda \to f(\lambda)\psi_\lambda$$



Auger effect

$$\psi(x,y) \qquad 0 \leqslant x, y \leqslant 2\pi$$

$$\langle \phi | \psi \rangle = \int_0^{2\pi}\!\!\int_0^{2\pi} \phi^*(x,y)\,\psi(x,y)\,dx\,dy$$

$$\hat{A} = -i\left(\frac{d}{dx} + \sqrt{2}\,\frac{d}{dy}\right) \quad ; \quad \psi(2\pi, y) = \psi(0, y)$$

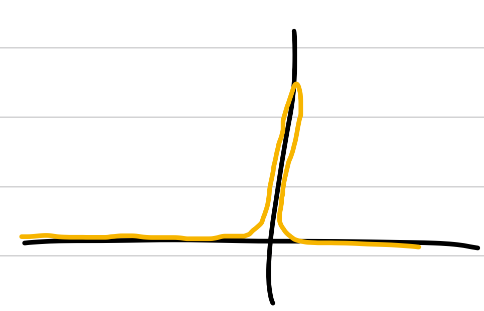
$$\psi(x, 2\pi) = \psi(x, 0)$$

$$\frac{e^{i(mx+ny)}}{2\pi} \quad ; \; m, n \in \mathbb{N} \qquad \lambda = m + \sqrt{2}\,n$$

Generalized functions

$$\hat{A}|\psi\rangle = \lambda|\psi\rangle$$

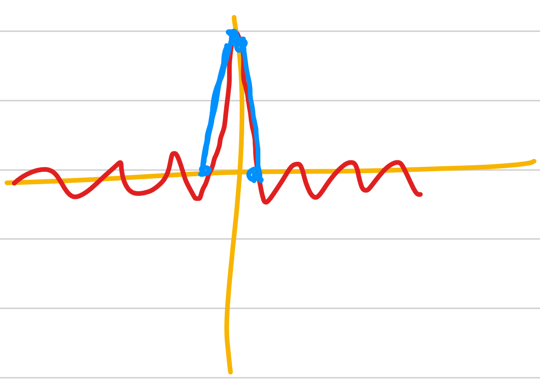
$$f(x) = \int_{-\infty}^{\infty} \delta(x-y)\, f(y)\, dy$$



$$v(x) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{m=-\infty}^{\infty} e^{im(x-y)} \, v(y)\, dy$$

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(x-y)} = \delta(x-y)$$

$$\frac{1}{2\pi}\sum_{m=-M}^{M} e^{imz} = \frac{1}{2\pi}\,\frac{\sin\left((2M+1)\,z/2\right)}{\sin(z/2)} \xrightarrow[M\gg 1]{} \frac{1}{2\pi}\,\frac{\sin(Mz) \sim Mz}{\sin(z/2) \to z/2}$$



Peak at $z=0$: $\frac{M}{\pi}$

nearest zero's: $\pm\frac{\pi}{M}$

$\Big\}$ Area: 1

another example: $$\sum_{m=0}^{\infty} (2m+1)\, P_m(x)\, P_m(y) = \delta(x-y)$$

$$\sum_{m=0}^{\infty} P_m(x) \underbrace{\int (2m+1)\, P_m(y)\, P_n(y)\, dy}_{\delta_{m,n}} = P_n(x)$$

$$\sum_{n=0}^{N} (2m+1)\, P_m(x)\, P_m(y) = \frac{(N+1)}{x-y}\Big(P_{n+1}(x)\, P_N(y) + P_N(x)\, P_{N+1}(y) \Big)$$

$$[q_n, p_m] = i\hbar\, \delta_{n,m}$$

$$\left.\begin{aligned} Q(x) &= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} q_m\, e^{imx} \\ P(x) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} p_n\, e^{-inx} \end{aligned}\right\} \quad \begin{aligned}[Q, P] &= \frac{i\hbar}{2\pi} \sum_m e^{im(x-y)} \\ &= i\hbar\, \delta(x-y)\end{aligned}$$

$$\gamma^\mu \underbrace{A_\mu \psi(x)}_{!}$$