Is local equilibrium a solution to Boltzmann equation?

$$\frac{\partial f_1^{(0)}}{\partial t} = \{H_1, f_1^{(0)}\} = -\vec{F}\cdot\frac{\partial f_1^{(0)}}{\partial \vec{p}} - \vec{v}\cdot\frac{\partial f_1^{(0)}}{\partial \vec{r}}$$

$$f_1^{(0)}(\vec{r},\vec{p},t) = n(\vec{r},t)\left(\frac{1}{2\pi m k T(\vec{r},t)}\right)^{3/2}\exp\left[-\frac{1}{2}\frac{m(\vec{v}-\vec{u}(\vec{r},t))^2}{kT(\vec{r},t)}\right]$$

$$* \quad \frac{\partial f_1^{(0)}}{\partial \vec{p}} = \frac{1}{m}\frac{\partial f^{(0)}}{\partial \vec{v}} = -\frac{1}{kT}(\vec{v}-\vec{u})\, f_1^{(0)}$$

$$D_t := \partial_t + \vec{u}\cdot\nabla$$

$$\tilde{D}_t := \partial_t + \vec{v}\cdot\nabla$$

$$* \quad \frac{\partial f_1^{(0)}}{\partial \vec{r}} = \frac{\partial f_1^{(0)}}{\partial n}\frac{\partial n}{\partial \vec{r}} + \frac{\partial f_1^{(0)}}{\partial T}\frac{\partial T}{\partial \vec{r}} + \frac{\partial f_1^{(0)}}{\partial \vec{u}}\cdot\frac{\partial \vec{u}}{\partial \vec{r}} \, *$$

$$\begin{cases} \dfrac{\partial f_1^{(0)}}{\partial n} = \dfrac{1}{n} f_1^{(0)} \\[2ex] \dfrac{\partial f_1^{(0)}}{\partial T} = -\dfrac{3}{2}\dfrac{1}{T} f_1^{(0)} + \dfrac{1}{2}\dfrac{m(\vec{v}-\vec{u})^2}{kT^2} f_1^{(0)} \\[2ex] \dfrac{\partial f^{(0)}}{\partial \vec{u}} = \dfrac{m}{kT}(\vec{v}-\vec{u})\, f_1^{(0)} \end{cases}$$

$$\tilde{D}_t f_1^{(0)} + \vec{F}\cdot\frac{\partial f_1^{(0)}}{\partial \vec{p}} = 0$$

$$\frac{\partial f_1^{(0)}}{\partial t} = \left[\frac{1}{n}\tilde{D}_t n + \left(\frac{m(\vec{v}-\vec{u})^2}{2kT^2} - \frac{3}{2T}\right)\tilde{D}_t T + \frac{m}{kT}(\vec{v}-\vec{u})\cdot\tilde{D}_t\vec{u} - \vec{F}\cdot\frac{(\vec{v}-\vec{u})}{kT}\right] f_1^{(0)}$$

$$\tilde{D}_t = D_t + (\vec{v}-\vec{u})\cdot\nabla$$

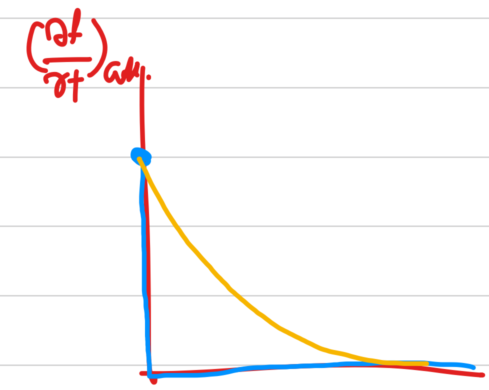
$$\begin{cases} D_t\rho + \rho\nabla\cdot\vec{u} = 0 \\ D_t T + \dfrac{2}{3}T\nabla\cdot\vec{u} = 0 \\ D_t\vec{u} + \dfrac{\nabla P}{\rho} = \dfrac{\vec{F}}{m} \end{cases}$$

$$+ P = nkT$$

$\longrightarrow \cdots \longrightarrow$

$$\frac{\partial f_1^{(0)}}{\partial t} - \{H_1, f_1^{(0)}\} = \Big[\frac{1}{T}\Big(\frac{m}{2kT}(\vec{v}-\vec{u})^2 - \frac{5}{2}\Big)(\vec{v}-\vec{u})\cdot\nabla T$$
$$+ \frac{m}{kT}\Big((v_i-u_i)(v_j-u_j) - \frac{1}{3}(\vec{v}-\vec{u})^2\,\delta_{ij}\Big) U_{ij} \Big] f_1^{(0)} \neq 0$$

$$f_1 = f_1^{(0)} + \underline{\underline{\delta f_1}}$$



Relaxation Time approximation :

$$\left(\frac{\partial f_1}{\partial t}\right)_{coll.} = \int d\Gamma\ \omega(p_1, p_2 | p_1', p_2') \big[f_1(p_1') f_1(p_2') - f_1(p_1) f_1(p_2) \big]$$
$$\simeq \int d\Gamma\ \omega(\cdots) \big[f_1^{(0)}(p_1')\,\delta f_1(p_2') + f_1^{(0)}(p_2')\,\delta f_1(p_1')$$
$$- f_1^{(0)}(p_1)\,\delta f_1(p_2) - f_1^{(0)}(p_2)\,\delta f_1(p_1) \big]$$

+ Chapman - Enskog expansion

$$\left(\frac{\partial f_1}{\partial t}\right)_{coll} = -\frac{\delta f_1}{\tau}$$ τ : Relaxation Time

$$f_1 = f_1^{(0)} + \delta f_1 \longrightarrow \frac{\partial}{\partial t}(f_1^{(0)} + \delta f_1) + \{H_1, f_1^{(0)} + \delta f_1\} = \left(\frac{\partial f_1}{\partial t}\right)_{coll.} - \frac{\delta f_1}{\tau}$$

$$\frac{\partial f_1^{(0)}}{\partial t} + \{H_1, f_1^{(0)}\} = -\frac{\delta f_1}{\tau}$$

$$\boxed{\delta f_1 = -\tau\left[\frac{1}{T}\left(\frac{m}{2kT}(\vec{v}-\vec{u})^2-\frac{5}{2}\right)(\vec{v}-\vec{u})\cdot\nabla T + \frac{m}{kT}\left((v_i-u_i)(v_j-u_j)-\frac{1}{3}(\vec{v}-\vec{u})^2\delta_{ij}\right)U_{ij}\right]f_1^{(0)}}$$

Thermal Conductivity:

$$q_i = \left\langle \rho(v_i-u_i)\ \tfrac{1}{2}m(\vec{v}-\vec{u})^2\right\rangle$$

$$= \tfrac{1}{2}\rho m\left\langle (v_i-u_i)(\vec{v}-\vec{u})^2\right\rangle$$

$f_1 = f_1^{(0)} + \delta f_1$ $\qquad \langle A\rangle = \int A f_1 \, d^3\vec{p}$

$$\hookrightarrow \langle A\rangle = \int A f_1^{(0)} d^3\vec{p} + \int A\,\delta f_1 \, d^3\vec{p} = \langle A\rangle_0 + \langle A\rangle_\delta$$

$$\tfrac{1}{2}\rho m\left\langle \underbrace{(v_i-u_i)(\vec{v}-\vec{u})^2}_{\text{odd}}\right\rangle_0 = 0$$

$\langle V-u\rangle = 0$

$\langle (V-u)^3\rangle = 0$

$$\delta f_1 = -\tau\left[\frac{1}{T}\left(\frac{m}{2kT}\underline{(\vec{v}-\vec{u})^2}-\frac{5}{2}\right)\underbrace{(\vec{v}-\vec{u})}_{\text{odd}}\cdot\nabla T + \frac{m}{kT}\left(\underbrace{(v_i-u_i)(v_j-u_j)}_{\text{even}}-\frac{1}{3}\underbrace{(\vec{v}-\vec{u})^2}\delta_{ij}\right)U_{ij}\right]f_1^{(0)}$$

$\langle (v-u)^2\rangle \neq 0$

$$\vec{q} = \frac{\rho m\tau}{2T}\int(\vec{v}-\vec{u})(\vec{v}-\vec{u})^2\left(\frac{m}{2kT}(\vec{v}-\vec{u})^2-\frac{5}{2}\right)(\vec{v}-\vec{u})\cdot\nabla T\, f_1^{(0)} d^3\vec{p}$$

$= K\,\nabla T$ $\qquad q_i = \sum_j k_{ij}\,\partial_j T$

$$K_{ij} = \frac{\rho m\tau}{2T}\int(v_i-u_i)(\vec{v}-\vec{u})^2\left(\frac{m}{2kT}(\vec{v}-\vec{u})^2-\frac{5}{2}\right)(v_j-u_j)\,f_1^{(0)} d^3\vec{p}$$

$$K_{ii} = \frac{\rho m\tau}{2T}\int(v_i-u_i)^2(\vec{v}-\vec{u})^2\left(\frac{m}{2kT}(\vec{v}-\vec{u})^2-\frac{5}{2}\right)f_1^{(0)}\, m\, d^3(\vec{v}-\vec{u})$$

$$K_{ii} = \frac{\rho m \tau}{2T} \int v_i^2 \frac{m}{2kT} v^4 - \frac{5}{2} v_i^2 v^2 f_1^{(0)} d^3\vec{p}$$

$$\langle v_i^2 \rangle = \frac{1}{3} \langle v^2 \rangle$$

$$K_{ii} = \frac{\rho m \tau}{6T} \left(\langle v^6 \rangle_0 - \frac{5}{2} \langle v^4 \rangle_0 \right)$$

$$\boxed{K_{ii} = \frac{5}{2} \tau n k_B^2 T}$$

$$C_p = \frac{5}{2} k_B \quad ; \quad \tau \sim \frac{1}{mn\sigma\sqrt{\langle v^2 \rangle}}$$

$$\langle v^2 \rangle \sim \frac{T}{m}$$

$$\rho\left(\partial_t + u_j \partial_j\right) kT + \frac{2}{3} \partial_j q_j + \frac{2}{3} m U_{ij} P_{ij} = 0$$

$$\vec{u} = 0 \quad , \nabla K = 0$$

$$\rho k \partial_t T + \frac{2}{3} K \nabla^2 T = 0$$

Viscosity :

$$P_{xy} = \rho \langle (v_x - u_x)(v_y - u_y) \rangle$$

$$P_{ij} = P \delta_{ij} + \pi_{ij}$$

π_{ij} . Stress Tensor

$$\pi_{ij} = \frac{m \tau \rho}{kT} U_{kl} \int d^3\vec{p} \, (v_i - u_i)(v_j - u_j)\left((v_k - u_k)(v_l - u_l) - \frac{1}{3}(\vec{v}-\vec{u})^2 \delta_{kl}\right) f_1^{(0)}$$

$$\Pi_{ij} = \frac{m\tau\rho}{kT} U_{kl} \left(\langle v_i v_j v_k v_l \rangle_0 - \frac{1}{3}\delta_{kl} \langle v_i v_j v^2 \rangle_0 \right)$$

$$\left\{ \begin{array}{l} \mathrm{Tr}\, \Pi_{ij} = 0 \\ \Pi_{ij} \to \text{linear with } U \end{array} \right.$$

$$\Pi_{ij} = \underbrace{\left(\Pi_{ij} - \frac{1}{3}\delta_{ij}\Pi_{kk} \right)}_{\text{Traceless}} + \underbrace{\frac{1}{3}\delta_{ij}\Pi_{kk}}_{\text{Trace}}$$

$$\Pi_{ij} = -2\eta \left(U_{ij} - \frac{1}{3}\delta_{ij} \nabla\cdot\vec{u} \right) \quad \longrightarrow \frac{\partial u_x}{\partial z} \neq 0$$

$$\Pi_{xz} = -2\eta\, U_{xz} = -2\eta \left(\frac{\cancel{\partial_x u_z} + \partial_z u_x}{2} \right) = -\eta \frac{\partial u_x}{\partial z} \quad \checkmark$$

$$\Pi_{xz} = \frac{m\tau\rho}{kT} U_{kl} \left(\langle v_x v_z v_k v_l \rangle_0 - \frac{1}{3}\delta_{kl} \langle v_x v_z v^2 \rangle_0 \right)$$

$$= \frac{m\tau\rho}{kT} \left(U_{zx} + U_{xz} \right) \langle v_x^2 v_z^2 \rangle$$

$$= \frac{m\tau\rho}{kT} \times \frac{2}{15}\, \partial_z u_x \langle v^4 \rangle \longrightarrow \eta = \frac{m\tau\rho}{kT} \times \frac{2}{15} \langle v^4 \rangle$$

$$\boxed{\eta = nkT\,\tau}$$

Navier Stokes :

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0}$$

Navier-Stokes

$\nabla \eta \simeq 0$

$$\boxed{(\partial_t + \vec{u}\cdot\nabla)\vec{u} = \frac{\vec{F}}{m} + \frac{\eta}{\rho}\left(\nabla^2\vec{u} + \frac{1}{3}\nabla(\nabla\cdot\vec{u}) \right) - \frac{\nabla P}{\rho}}$$

$\nabla K = 0$

$U_{ij}\Pi_{ij} \simeq 0$

$$\boxed{\rho D_t T = \frac{2}{3} K \nabla^2 T + \frac{2}{3} mp\, \nabla\cdot\vec{u}}$$