Master Equation:

$$\frac{\partial}{\partial t}\langle nA\rangle+\frac{\partial}{\partial \vec r}\cdot\langle n\vec v A\rangle - n\langle \vec v\cdot\frac{\partial A}{\partial \vec r}\rangle - n\langle \vec F\cdot\frac{\partial A}{\partial \vec p}\rangle = 0$$

$$\partial_t = \frac{\partial}{\partial t} \qquad \partial_i = \frac{\partial}{\partial r_i}$$

Density: $A=1$: $\partial_t n + \nabla\cdot(n\vec u) = 0$; $\rho := mn$, $\vec u := \langle \vec v\rangle$

$$\boxed{\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec u)=0} \rightarrow \text{Continuity eqn} \qquad \vec J := \rho\vec u$$

$$\partial_t\rho + \partial_j(\rho u_j) = \underbrace{\partial_t\rho + \rho\,\partial_j u_j + u_j\partial_j\rho = 0}$$

$A(\vec r,\vec p)$

Momentum: $A = p_i = mv_i$

$$\frac{\partial}{\partial t}\langle nA\rangle+\frac{\partial}{\partial \vec r}\cdot\langle n\vec v A\rangle - \cancel{n\langle \vec v\cdot\frac{\partial A}{\partial \vec r}\rangle} - n\langle \vec F\cdot\frac{\partial A}{\partial \vec p}\rangle = 0 \qquad \frac{\partial}{\partial p_j}A = \delta_{ij}$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial r_j}\Big(\rho\,\underbrace{\langle v_j v_i\rangle}_{?}\Big) - nF_i = 0$$

Pressure Tensor: $\rho\big\langle (v_i-u_i)(v_j-u_j)\big\rangle =: P_{ij} = P_{ji}$ $\rightarrow$ flux of i-th momentum in j-th direction

Maxwell-Boltzmann $\rightarrow$ $P_{ij} = nkT\,\delta_{ij} = P$ — flux of momentum $\leftrightarrow$ $\frac{\text{force}}{\text{Area}}$ $\leftrightarrow$ Pressure

$\langle v_j u_i\rangle = u_i\underbrace{\langle v_j\rangle}_{u_j}$

$$\rho\langle v_j v_i\rangle = \rho\langle (v_j-u_j)(v_i-u_i)\rangle + \underbrace{\rho\langle v_j u_i\rangle}_{\rho u_i u_j} + \underbrace{\rho\langle v_i u_j\rangle}_{\rho u_i u_j} - \rho u_i u_j$$

$$= P_{ij} + \rho u_i u_j$$

$$\partial_t(\rho u_i) + \partial_j(P_{ij}+\rho u_i u_j) - nF_i = 0$$

$$u_i\partial_t\rho + \rho\,\partial_t u_i + \partial_j P_{ij} + \rho u_i\partial_j u_j + \rho u_j\partial_j u_i + u_i u_j\partial_j\rho - nF_i = 0$$

Continuity: $\partial_t\rho + \rho\,\partial_j u_j + u_j\partial_j\rho = 0$

$$\boxed{\rho\left(\partial_t + u_j\partial_j\right)u_i = -\partial_j P_{ij} + \frac{\rho}{m}F_i}$$ $D_t \equiv \partial_t + \vec{u}\cdot\nabla$

$$\rho D_t u_i = -\partial_j P_{ij} + \frac{\rho}{m}F_i$$ $\longrightarrow$ Newton's second law!

Kinetic energy : $A = \frac{1}{2}m\left(\vec{v}-\vec{u}\right)^2$

$$\frac{1}{2}\left\langle F_j \overbrace{\frac{\partial}{\partial v_j}(\vec{v}-\vec{u})^2}^{2(v_j-u_j)}\right\rangle \qquad F_j\left\langle (v_j-u_j)\right\rangle = 0$$

$$\sum_i \frac{\partial}{\partial_j}\left(v_i-u_i\right)^2 = \sum_i 2(v_i-u_i)\frac{\partial u_i}{\partial r_j}$$

$$\longrightarrow \frac{\partial}{\partial t}\langle nA\rangle + \frac{\partial}{\partial \vec{r}}\cdot\langle n\vec{v}A\rangle - n\left\langle \vec{v}\cdot\frac{\partial A}{\partial \vec{r}}\right\rangle - n\overbrace{\left\langle \vec{F}\cdot\frac{\partial A}{\partial \vec{p}}\right\rangle}=0 \quad , \quad \partial_j(\vec{v}-\vec{u})^2 = 2(v_i-u_i)\partial_j u_i$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left\langle \rho(\vec{v}-\vec{u})^2\right\rangle + \frac{1}{2}\frac{\partial}{\partial r_j}\left\langle \rho v_j(\vec{v}-\vec{u})^2\right\rangle - \rho\left\langle \underbrace{\frac{v_j}{2}\frac{\partial}{\partial r_j}(\vec{v}-\vec{u})^2}_{v_j\partial_j u_i(v_i-u_i)}\right\rangle = 0$$

$$\longrightarrow \frac{1}{2}\partial_t\left\langle \rho(\vec{v}-\vec{u})^2\right\rangle + \frac{1}{2}\partial_j\left\langle \rho v_j(\vec{v}-\vec{u})^2\right\rangle - \rho\left\langle v_j\partial_j u_i(v_i-u_i)\right\rangle = 0$$

$*$ $A(\vec{r},\vec{p})$ in master equation is time independent while this one is time dependent through u $*$

extra term $\sim \left\langle (\vec{v}-\vec{u})\cdot\frac{\partial\vec{u}}{\partial t}\right\rangle = \left\langle \vec{v}-\vec{u}\right\rangle\cdot\frac{\partial\vec{u}}{\partial t} = 0$ ✓

Temperature : $\frac{3}{2}kT(\vec{r},t) = \frac{1}{2}m\left\langle (\vec{v}-\vec{u}(\vec{r},t))^2\right\rangle$ $*$ $\mathrm{Tr}\,P = 3\rho\frac{kT}{m}$ $*$

Heat flux : $q_i := \frac{1}{2}m\rho\left\langle (v_i-u_i)(v_j-u_j)^2\right\rangle$

$$\frac{1}{2}m\rho\left\langle v_i(\vec{v}-\vec{u})^2\right\rangle = \frac{1}{2}m\rho\left\langle (v_i-u_i)(\vec{v}-\vec{u})^2\right\rangle + \frac{1}{2}m\rho\, u_i\left\langle (v_j-u_j)^2\right\rangle$$

$$= q_i + \frac{3}{2}\rho kT u_i$$

$$\rho\left\langle v_j\partial_j u_i(v_i-u_i)\right\rangle = \rho\,\partial_j u_i\left(\left\langle (v_j-u_j)(v_i-u_i)\right\rangle + u_j\overbrace{\left\langle v_i-u_i\right\rangle}^{0}\right)$$

$$= \partial_j u_i \quad P_{ij}$$

$$\frac{3}{2}\frac{\partial}{\partial t}(\rho k T) + \frac{\partial}{\partial r_j}\left(q_i + \frac{3}{2}\rho k T u_i\right) + m P_{ij}\,\partial_j u_i = 0$$

Rate of strain : $U_{ij} := \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$

+ Continuity

$$\boxed{\rho\left(\partial_t + u_j\partial_j\right) k T + \frac{2}{3}\partial_j q_j + \frac{2}{3} m\, U_{ij} P_{ij} = 0}$$

$\rho, \vec{u}, P_{ij}, q$ (!) $\qquad f_1 = ???$ $\longleftrightarrow$ Boltzmann Eq.

Boltzmann Equation is hard !

Ideal Fluid : what about heat equilibrium?

$$f_1^{(0)}(\vec{r}, \vec{p}, t) = n(\vec{r}, t)\left(\frac{1}{2\pi m k T(\vec{r}, t)}\right)^{3/2} \exp\left[-\frac{1}{2}\frac{m\left(\vec{v} - \vec{u}(\vec{r}, t)\right)^2}{k T(\vec{r}, t)}\right]$$

find P_{ij} & q_i : $P_{ij} = \rho\left\langle (v_i - u_i)(v_j - u_j)\right\rangle = n k T\, \delta_{ij} \equiv P(\vec{r}, t)\, \delta_{ij}$

$$q_i = \frac{1}{2} m \rho \left\langle (v_i - u_i)(\vec{v} - \vec{u})^2\right\rangle = 0 \quad !$$

Equations of motion : – Continuity : $\partial_t \rho + \nabla\cdot(\rho\vec{u}) = 0$

$$\hookrightarrow \left(\partial_t + u_j\partial_j\right)\rho + \rho\,\partial_j u_j = 0$$

– momentum : Euler equation

$$\left(\partial_t + u_j\partial_j\right) u_i + \frac{1}{\rho}\partial_i P = \frac{F_i}{m}$$

Energy . flow of heat

$$\left(\partial_t + u_j \partial_j\right) T + \frac{2}{3} T \partial_j u_j = 0$$

Ideal fluid $\longrightarrow$ they miss dissipation !

$\hookrightarrow$ no irreversibility $\longrightarrow$ no equilibrium !

$f_1^{(0)} \longrightarrow$ Detailed Balance $\longrightarrow \frac{dH}{dt} = 0 \longrightarrow$ no increase of entropy !

Directly :

$$D_t \rho = -\rho \partial_i u_i$$

$$D_t T = -\frac{2}{3} T \partial_i u_i$$

$$D_t\left(\rho T^{-3/2}\right) = T^{-3/2} D_t \rho - \frac{3}{2} T^{-5/2} \rho D_t T = -\rho T^{-3/2} \partial_i u_i + T^{-3/2} \rho \partial_i u_i = 0$$

$\rho T^{-3/2} = \text{Constant} \longrightarrow$ Adiabatic Process !

$q_i = 0$