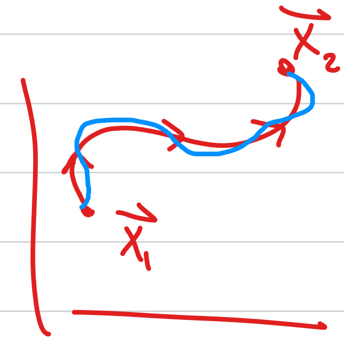
Lagrangian: S : action $\qquad \vec{X}_1 = (t_1, \vec{q}, \dot{\vec{q}})$

$$S(\vec{X}_1, \vec{X}_2) = \int_{\vec{X}_1}^{\vec{X}_2} L(\vec{q}, \dot{\vec{q}}; t)\, dt$$



q

$q' = q + \delta q$

$$\delta S = 0 \longrightarrow \delta S = \int \delta L \, dt \quad ; \quad q_0 + \delta q$$

$$\delta L = \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i$$

$$\delta(\dot{q}_i) = \dot{q}_i' - \dot{q}_i = \frac{d}{dt}(q_i' - q_i) = \frac{d}{dt}(\delta q_i)$$

$$\frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i = \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}(\delta q_i) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i$$

$$\delta L = \sum_i \frac{\partial L}{\partial q_i} \delta q_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i\right) \qquad \int \frac{d}{dt}(\ ) \, dt$$

$$\delta S = \int \delta L \, dt = \cancel{\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{\vec{X}_1}^{\vec{X}_2}} + \int_{\vec{X}_1}^{\vec{X}_2} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right) \delta q_i \, dt = 0 \quad ; \ \forall \, \delta q_i$$

$$\Longrightarrow \boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad \forall \, i}$$

Coordinates Transformation:

$$L(x, \dot{x}, t) \ , \ L(q, \dot{q}, t) \qquad x_i = x_i(q, t)$$

$$x_i = x_i(q, t) \longrightarrow \dot{x}_j = \sum_k \frac{\partial x_j}{\partial q_k} \dot{q}_k + \frac{\partial x_j}{\partial t}$$

$$\frac{\partial L}{\partial q_i} = \sum_j \frac{\partial L}{\partial x_j} \frac{\partial x_j}{\partial q_i} + \frac{\partial L}{\partial \dot{x}_j} \frac{\partial \dot{x}_j}{\partial q_i} \quad ; \quad \frac{\partial \dot{x}_j}{\partial q_i} = \sum_k \frac{\partial^2 x_j}{\partial q_i \partial q_k} \dot{q}_k + \frac{\partial^2 x_j}{\partial t \partial q_i}$$

$$\frac{\partial L}{\partial q_i} = \sum_j \frac{\partial L}{\partial x_j}\frac{\partial x_j}{\partial q_i} + \frac{\partial L}{\partial \dot{x}_j}\left(\sum_k \frac{\partial^2 x_j}{\partial q_i \partial q_k}\dot{q}_k + \frac{\partial^2 x_j}{\partial t\, \partial q_i}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) :$$

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_j \frac{\partial L}{\partial x_j}\underbrace{\frac{\partial x_j}{\partial \dot{q}_i}}_{0} + \frac{\partial L}{\partial \dot{x}_j}\underbrace{\frac{\partial \dot{x}_j}{\partial \dot{q}_i}}$$

$$\dot{x}_j = \sum_k \frac{\partial x_j}{\partial q_k}\dot{q}_k + \frac{\partial x_j}{\partial t} \longrightarrow \frac{\partial \dot{x}_j}{\partial \dot{q}_i} = \frac{\partial x_j}{\partial q_i}$$

$$\frac{\partial L}{\partial \dot{q}_i} = \sum_j \frac{\partial L}{\partial \dot{x}_j}\frac{\partial x_j}{\partial q_i} \quad ; \quad \frac{d}{dt} = \sum_k \dot{q}_k \frac{\partial}{\partial q_k} + \ddot{q}_k \frac{\partial}{\partial \dot{q}_k} + \frac{\partial}{\partial t}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right)\frac{\partial x_j}{\partial q_i} + \frac{\partial L}{\partial \dot{x}_j}\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right)$$

$$\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_i}\right) = \sum_k \dot{q}_k \frac{\partial^2 x_j}{\partial q_i \partial q_k} + \ddot{q}\,\cancel{\frac{\partial^2 x_j}{\partial q_i \partial \dot{q}_k}} + \frac{\partial^2 x_j}{\partial t\, \partial q_i}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right)\frac{\partial x_j}{\partial q_i} + \frac{\partial L}{\partial \dot{x}_j}\left(\sum_k \dot{q}_k \frac{\partial^2 x_j}{\partial q_i \partial q_k} + \frac{\partial^2 x_j}{\partial t\, \partial q_i}\right)$$

$$\frac{\partial L}{\partial q_i} = \sum_j \frac{\partial L}{\partial x_j}\frac{\partial x_j}{\partial q_i} + \frac{\partial L}{\partial \dot{x}_j}\left(\sum_k \frac{\partial^2 x_j}{\partial q_i \partial q_k}\dot{q}_k + \frac{\partial^2 x_j}{\partial t\, \partial q_i}\right)$$

$$\boxed{\sum_j \frac{\partial x_j}{\partial q_i}\left(\frac{\partial L}{\partial x_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_j}\right)\right) = \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)}$$

$$\boxed{\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0 \iff \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0}$$

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i} \quad ; \quad L = T - V = \sum_i \frac{1}{2} m \dot{x}_i^2 - V$$

$$\frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i}$$

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i \rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = m\ddot{x}_i$$

Legendre Transformation :

$$dU = TdS - PdV \qquad U(S,V) \rightarrow T = \frac{\partial U}{\partial S}, \; P = -\frac{\partial U}{\partial V}$$

$$= d(TS) - SdT - PdV$$

$$d(\underbrace{U - TS}_{F}) = -SdT - PdV \rightarrow dF = -SdT - PdV \; ; \; F(T,V)$$

$$S = -\frac{\partial F}{\partial T}, \; P = -\frac{\partial F}{\partial V}$$

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \quad , \quad \frac{\partial L}{\partial \dot{q}_i} =: P_i \qquad \frac{\partial L}{\partial q_i} = \dot{P}_i$$

$$dL = \sum_i \dot{P}_i dq_i + P_i d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

$$dL = \sum_i \dot{P}_i dq_i + d(P_i \dot{q}_i) - \dot{q}_i dP_i + \frac{\partial L}{\partial t} dt$$

$$d(\underbrace{L - \sum_i P_i \dot{q}_i}_{-H}) = \sum_i \dot{P}_i dq_i - \dot{q}_i dP_i + \frac{\partial L}{\partial t} dt \; ; \; H := \sum_i P_i \dot{q}_i - L$$

$$dH = \sum_i \dot{q}_i dP_i - \dot{P}_i dq_i - \frac{\partial L}{\partial t} dt$$

$$\boxed{\frac{\partial H}{\partial p_i} = \dot{q}_i} \; , \; \boxed{\frac{\partial H}{\partial q_i} = -\dot{p}_i} \; , \; \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = \frac{dH}{dt}$$

$$\frac{dH}{dt} = \sum_i \dot{q}_i \dot{p}_i - \dot{p}_i \dot{q}_i \; - \frac{\partial L}{\partial t} = -\frac{\partial L}{\partial t}$$

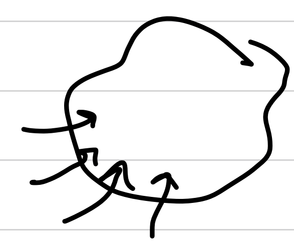
$$A(q,p) \; . \quad \frac{d}{dt} A(q,p) = \frac{\partial A}{\partial t} + \sum_i \frac{\partial A}{\partial q_i} \dot{q}_i + \frac{\partial A}{\partial p_i} \dot{p}_i$$

$$= \frac{\partial A}{\partial t} + \sum_i \frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i}$$

Poisson Bracket. $\{f, g\} := \sum_i \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i}$

$$\frac{d}{dt} A = \frac{\partial A}{\partial t} + \{A, H\}$$

Fluid dynamics: Conservation of mass . $\frac{d}{dt}\left(\int \rho \, d\bar{V}\right) = -\oint \vec{J} \cdot \vec{dS}$



$\vec{J} = \rho \vec{V}$ $\qquad \int_V \frac{\partial \rho}{\partial t} \, d\bar{V} = -\oint_{\partial V} \rho \vec{V} \cdot \vec{dS}$

$$\int \rho \vec{V} \cdot \vec{dS} = \int \nabla \cdot (\rho \vec{V}) \, d\bar{V} \Longrightarrow \int \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right) d\bar{V} = 0 \quad \forall \bar{V}$$

$$\hookrightarrow \underline{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0}$$

Conservation

<u>incompressibility</u>: $\frac{d\rho}{dt} = 0 \quad ; \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \sum_i \dot{x}_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial t} + \vec{V} \cdot \nabla$

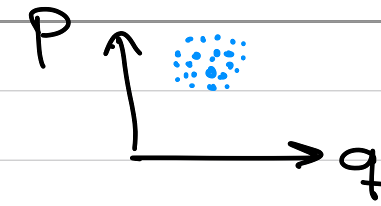
$$\hookrightarrow \underline{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho = 0}$$

incompressibility

$$\nabla \cdot (\rho \vec{V}) = \nabla \rho \cdot \vec{V} + \rho \nabla \cdot \vec{V} \xrightarrow[\text{Conservation}]{} \underbrace{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \nabla \rho}_{\text{incompressibility} = 0} + \rho \nabla \vec{V} = 0$$

$$\boxed{\nabla \cdot \vec{V} = 0}$$

Phase space : N – Particles



$\hookrightarrow$ 6N – dimensional

$$\vec{r} := (q_1, q_2, \dots, p_1, p_2, \dots)^T$$

$$\vec{v} = \frac{d\vec{r}}{dt} = (\dot{q}_1, \dot{q}_2, \dots, \dot{p}_1, \dot{p}_2, \dots)^T$$

$$\nabla \equiv \sum_{i=1}^{3N} \hat{q}_i \frac{\partial}{\partial q_i} + \hat{p}_i \frac{\partial}{\partial p_i} \qquad \frac{\partial H}{\partial p_i} = \dot{q}_i \ , \ -\frac{\partial H}{\partial q_i} = \dot{p}_i$$

$$\nabla \cdot \vec{v} = \sum_{i=1}^{3N} \frac{\partial}{\partial q_i} \dot{q}_i + \frac{\partial}{\partial p_i} \dot{p}_i = \sum_i \frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} = 0$$

$\underline{\underline{\nabla \cdot \vec{v} = 0}} \longrightarrow$ incompressible fluid ! $\longrightarrow$ liouville's thm

$$\rho := \frac{dn}{d^{3N}q \, d^3p} \quad ; \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{v} \cdot \nabla \rho = 0$$

$$\frac{\partial \rho}{\partial t} + \sum_i \dot{q}_i \frac{\partial \rho}{\partial q_i} + \dot{p}_i \frac{\partial \rho}{\partial p_i} = 0$$

$$\frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i} \frac{\partial \rho}{\partial p_i} = 0$$

$$\frac{\partial \rho}{\partial t} + \{\rho, H\} = 0 \longrightarrow \frac{d\rho}{dt} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} = \{H, \rho\}} \quad : \text{Liouville's eqn.}$$

Poincaré 's thm :

- Bounded Phase space
- Eqns. of motion are reversible