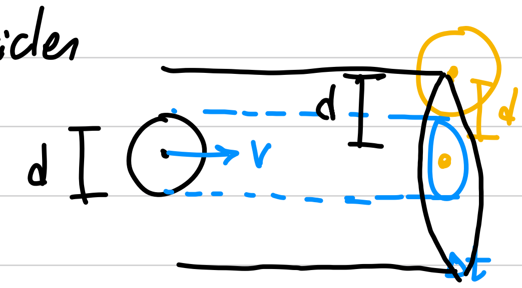
Transport Phenomena :

1. mean free path

n : density of particles



$$\Delta V = v\,\Delta t\;\pi d^2$$

$$\langle \Delta V\rangle = \bar{v}\,\Delta t\;\pi d^2 \;\rightarrow\; \langle N\rangle = n\,\bar{v}\,\Delta t\;\pi d^2$$

Rate of Collisions : $n\bar{v}\,\pi d^2 = f$: $\tau = \frac{1}{f}$

$$\tau = \frac{1}{\pi d^2 n \bar{v}} \rightarrow \lambda = \bar{v}\,\tau = \boxed{\frac{1}{\pi d^2 n} = \lambda}$$ due to clausius

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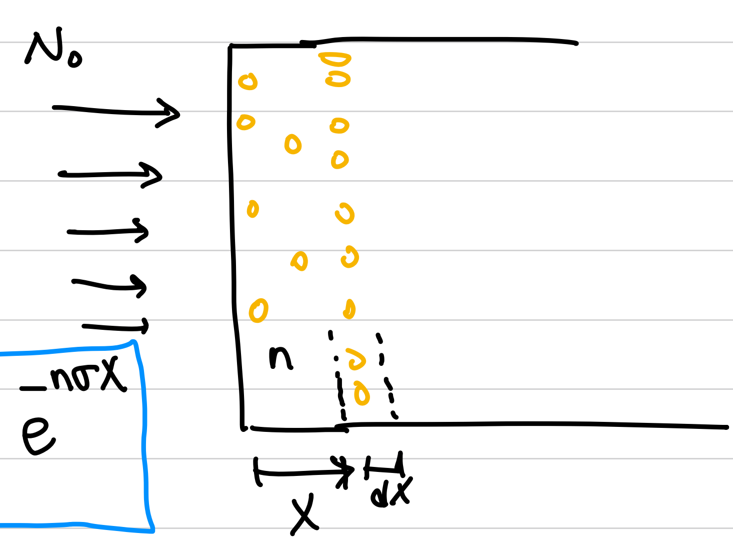
σ : total scattering Cross-section



$$-\frac{dN}{N} = \frac{n A dX\,\sigma}{A} \rightarrow \frac{dN}{N} = -n\sigma\,dX$$

$$\ln\left(\frac{N}{N_0}\right) = -n\sigma X \rightarrow \boxed{N(X) = N_0\,e^{-n\sigma X}}$$

$$\lambda = \frac{x_1\,\Delta N_1 + x_2\,\Delta N_2 + \cdots}{\Delta N_1 + \Delta N_2 + \cdots} = \frac{\int x\,dN}{N_0} = \frac{1}{N_0}\int^{\infty} x\left(-N_0\, n\sigma\, e^{-n\sigma X}\right) dX$$

$$= -n\sigma \int_0 x\, e^{-n\sigma X}\, dX = \frac{1}{n\sigma}$$

$$\boxed{\lambda = \frac{1}{n\sigma}}$$

More Precise : $\tau = \frac{1}{n\sigma \bar{V}_{rel}}$ $\qquad \bar{v} \rightarrow \bar{V}_{rel}$

$$\vec{V}_{rel} = \vec{V}_2 - \vec{V}_1 \rightarrow V_{rel}^2 = V_1^2 + V_2^2 - 2\vec{V}_1\cdot\vec{V}_2 \rightarrow \langle V_{rel}^2\rangle = 2\langle v^2\rangle - 2\langle \vec{V}_1\cdot\vec{V}_2\rangle$$

$$\langle \vec{V}_1\cdot\vec{V}_2\rangle = \iint \vec{V}_1\cdot\vec{V}_2\, f(\vec{V}_1)\, f(\vec{V}_2)\, d^3\vec{V}_1\, d^3\vec{V}_2 = \int f(\vec{V}_1)\,\vec{V}_1\cdot\Big(\underbrace{\int \vec{V}_2\, f(\vec{V}_2)\, d^3\vec{V}_2}_{0}\Big)\, d^3\vec{V}_1$$

$$\langle \vec{V}_1 \cdot \vec{V}_2 \rangle = 0 \longrightarrow \langle V_{rel}^2 \rangle = 2 \langle V^2 \rangle \longrightarrow \bar{V}_{rel} = \sqrt{2}\,\bar{V}$$

$$\tau = \frac{1}{n\sigma\sqrt{2}\,\bar{V}} \longrightarrow \boxed{\lambda = \bar{V}\tau = \frac{1}{n\sigma\sqrt{2}}}$$

$$PV = NkT \longrightarrow P = nkT \longrightarrow n = \frac{P}{kT} \longrightarrow \lambda = \frac{1}{\sigma\sqrt{2}}\left(\frac{kT}{P}\right)$$

oxygen: $d = 3.6\,\text{Å} \longrightarrow \underline{\lambda \sim 10^{-8}\ \text{m}}$

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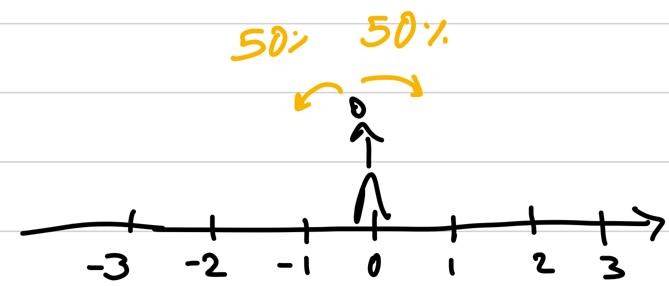
2. Diffusion:

Random walk: $P(m, N)$: Prob. of being at m after N steps

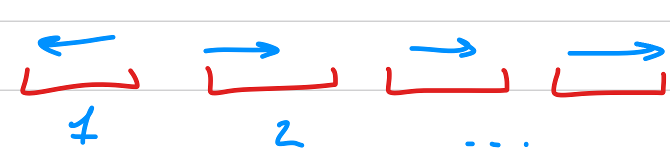


N_r: # of steps to the right

N_l: # " " " left

$$\begin{cases} N_r + N_l = N \\ N_r - N_l = m \end{cases}$$

$$N_r = \frac{N+m}{2}, \quad N_l = \frac{N-m}{2}$$



$$\left(\tfrac{1}{2}\right)^N + \left(\tfrac{1}{2}\right)^N + \left(\tfrac{1}{2}\right)^N$$

$$\binom{N}{N_r} = \frac{N!}{N_r!\,(N-N_r)!};$$

$$P(m, N) = \binom{N}{N_r}\left(\frac{1}{2}\right)^N = 2^{-N}\binom{N}{N_r}$$

$$P(m,N) = \frac{2^{-N}\, N!}{N_r!\; N_l!} \quad ; \quad N_r = \frac{N+m}{2} \, , \; N_l = \frac{N-m}{2}$$

$$\begin{cases} N \gg 1 \\ N \gg m \end{cases} \; : \qquad \ln(N!) \simeq N\ln N - N + \frac{1}{2}\ln(2\pi N)$$

$$\ln\left(\frac{N+m}{2}\right) = \ln\left(\frac{N}{2}\left(1+\frac{m}{N}\right)\right) \simeq \ln\left(\frac{N}{2}\right) + \frac{m}{N}$$

$$\ln\binom{N}{N_r} \simeq N\ln N - N_r\ln N_r - N_l\ln(N_l) + \frac{1}{2}\ln\left(\frac{N}{2\pi N_r N_l}\right)$$

$$\left.\begin{aligned} N_r &= \frac{N+m}{2} \\ N_l &= \frac{N-m}{2} \end{aligned}\right\} \ln\binom{N}{N_r} \simeq N\ln N - \left(\frac{N+m}{2}\right)\left(\ln\frac{N}{2} + \frac{m}{N}\right) - \left(\frac{N-m}{2}\right)\left(\ln\frac{N}{2} - \frac{m}{N}\right) + \frac{1}{2}\ln\left(\frac{2N}{\pi(N^2-m^2)}\right)$$

$$\begin{cases} \mathcal{O}(0) = N\ln(2) + \frac{1}{2}\ln\left(\frac{2}{\pi N}\right) \\ \mathcal{O}(1) = 0 \\ \mathcal{O}(2) = -\frac{m^2}{2N} \end{cases}$$

$m, N \longrightarrow x, t$

each step : l , τ

$\hookrightarrow m = \frac{x}{l}$, $N = \frac{t}{\tau}$

$$2^{-N}\binom{N}{N_r} = \sqrt{\frac{2}{\pi N}}\, e^{-\frac{m^2}{2N}} = \boxed{\sqrt{\frac{2\tau}{\pi t}}\, e^{-\frac{x^2\tau}{2l^2 t}}} = P(x,t)$$

$$\langle x \rangle = 0 \quad ; \quad \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x,t)\, dx = \frac{l^2}{\tau}\, t$$

$$\sqrt{\langle x^2 \rangle} \sim \sqrt{t}$$

3D : $\tau \to 3\tau$: $\langle x^2 \rangle = \frac{l^2}{3\tau} t$

$$\langle \vec{r}^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle = 3 \langle x^2 \rangle = \frac{l^2}{\tau} t$$

Diffusion : $n(t+\Delta t, X) = \sum_{X'} n(t, X') P(X-X', \Delta t)$

$\Delta X := X' - X$

$$= \int n(t, X') P(X-X', \Delta t) \, dX'$$

$$= \int n(t, X+\Delta X) P(\Delta X, \Delta t) \, d(\Delta X)$$

$$= \langle n(t, X+\Delta X) \rangle$$

$$= \left\langle n(t,X) + \frac{\partial n}{\partial X} \Delta X + \frac{\partial^2 n}{\partial X^2} \Delta X^2 + \cdots \right\rangle$$

$$n(t+\Delta t, X) = n(t,X) + \frac{\partial n}{\partial X} \underbrace{\langle \Delta X \rangle}_{0} + \frac{1}{2} \frac{\partial^2 n}{\partial x^2} \langle \Delta X^2 \rangle$$

$$\boxed{\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial X^2}} \quad ; D = \frac{\langle \Delta X^2 \rangle}{2 \Delta t}$$

Solution to Diffusion eq. :

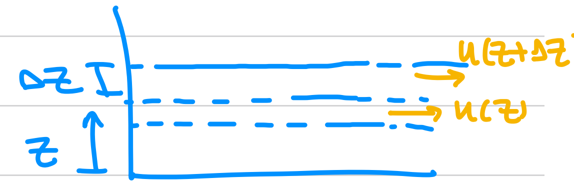
$$n(0, X) = N \delta(X) \to N(X,t) = N \sqrt{\frac{1}{4\pi D t}} e^{-\frac{X^2}{4Dt}}$$

General sol. $n(0, X) = n_0(X) \longrightarrow$ $\boxed{N(X,t) = \int_{-\infty}^{\infty} \sqrt{\frac{1}{4\pi D t}} e^{-\frac{(X-X')^2}{4Dt}} n_0(X') \, dX'}$

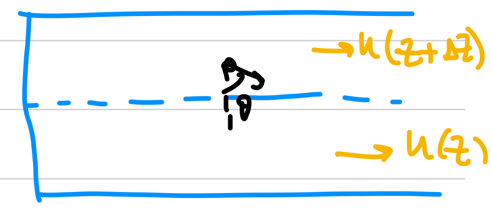
3D : $\frac{\partial n}{\partial t} = D \nabla^2 n$

Viscosity :

$$F = \eta A \frac{du}{dz} \longrightarrow \text{Empirical}$$



$$\Delta P = m\left(u(z+\Delta z) - u(z)\right) = m \frac{du}{dz}\Delta z$$



$$F = -\frac{\Delta P}{\Delta t} \longrightarrow \frac{F}{A} = -\frac{1}{A}\frac{\Delta P}{\Delta t}$$

# of Particles per unit area: $n v_z f(\vec{v})\, d^3\vec{v}$

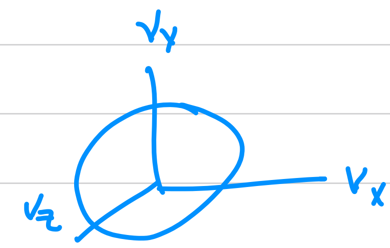
$$\frac{1}{A}\frac{\Delta P}{\Delta t} = \int n v_z f(\vec{v})\, m \frac{du}{dz}\Delta z\, d^3\vec{v} \quad ; \quad \Delta z = l\cos\theta, \; v_z = v\cos\theta$$

$d^3\vec{v} = v^2 \sin\theta\, dv\, d\theta\, d\phi$

$$= \left(m\frac{du}{dz} l\right) 2\pi n \left(\int v^3 f(v)\, dv\right) \underbrace{\left(\int_0^{\pi} \cos^2\theta \sin\theta\, d\theta\right)}_{\frac{2}{3}}$$

$\sin\theta\, d\theta = -d(\cos\theta)$

$$= \frac{1}{3} mnl \frac{du}{dz} \underbrace{\int v f(v)\, 4\pi v^2\, dv}_{\bar{v}}$$



$$\bar{v} = \int v f(v) \underbrace{d^3\vec{v}}_{\underbrace{\sin\theta\, d\theta\, d\phi\, v^2 dv}_{4\pi}}$$

$$= \frac{1}{3} mnl\bar{v} \frac{du}{dz}$$

$$\boxed{\eta = \frac{1}{3} mnl\bar{v}}$$

$$\vec{q} = K\nabla T \quad ; \quad E = \frac{3}{2}kT \longrightarrow \Delta E = \frac{3}{2}k\frac{\partial T}{\partial z}\overbrace{\Delta z}^{l\cos\theta}$$

$$|\vec{q}| = n\int d^3\vec{v}\, \Delta E\, v_z f(\vec{v}) \longrightarrow \cdots \longrightarrow |\vec{q}| = \frac{1}{2}knl\bar{v}\frac{\partial T}{\partial z}$$

$$\boxed{K = \frac{1}{3} c_v l \bar{v}}, \quad \boxed{c_v = \frac{3}{2} nk}$$