Note: Convergence in the Continuous Spectrum

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August 25, 2025

Abstract

In this note, I will show that although continuous operators do not have eigenvectors in the Hilbert space, it is possible to define converging sequences for them in the Hilbert space.

1 Initial Definitions

I will work with the operator under consideration. For simplicity, and without reducing the generality of the problem, I consider the position operator. I define the operator $\hat{\mathbb{E}}(\xi)$ as follows:

$$\hat{\mathbb{E}}(\xi) := \begin{cases} 1 & x < \xi \\ 0 & x > \xi \end{cases}$$

Thus,

$$\mathrm{d}\hat{\mathbb{E}}\,(\xi) = \hat{\mathbb{E}}(\xi + \mathrm{d}\xi) - \hat{\mathbb{E}}(\xi)$$

This limit is well-defined and bounded. Additionally, this operator satisfies the following relations:

$$\hat{\mathbb{E}}(x_{\min}) = 0 \qquad , \qquad \hat{\mathbb{E}}(x_{\max}) = \mathbb{1}$$

$$\int_0^{\mathbb{1}} \mathrm{d}\hat{\mathbb{E}}\,(\xi) = \hat{\mathbb{1}}$$

$$\hat{\mathbb{E}}(\xi)\hat{\mathbb{E}}(\xi') = \hat{\mathbb{E}}(\xi')\hat{\mathbb{E}}(\xi) = \begin{cases} \hat{\mathbb{E}}(\xi) & \xi \leq \xi' \\ \hat{\mathbb{E}}(\xi') & \xi \geq \xi' \end{cases} \tag{1}$$

The position operator (which is x in position representation) can be written as:

$$x = \int_0^{\mathbb{1}} \xi\,\mathrm{d}\hat{\mathbb{E}}\,(\xi)$$

And similarly:

$$f(x) = \int_0^{\mathbb{1}} f(\xi)\,\mathrm{d}\hat{\mathbb{E}}\,(\xi)$$

From equation (1), we can deduce:

$$\left(\int f(\xi)\,\mathrm{d}\hat{\mathbb{E}}\,(\xi)\right)\left(\int g(\eta)\,\mathrm{d}\hat{\mathbb{E}}\,(\eta)\right) = \int f(\sigma)g(\sigma)\,\mathrm{d}\hat{\mathbb{E}}\,(\sigma) \tag{2}$$

2 Strong Convergence to Eigenvectors

Consider any desired ε. I define the following operator:

$$\hat{P}_\lambda := \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \mathrm{d}\hat{\mathbb{E}}\,(\xi)$$

It can be easily shown that this is a projection operator. Also, assume that ψ_λ is the eigenvector of this operator with eigenvalue 1. Constructing such a vector is very simple. It is enough to find a vector such as ϕ such that $\hat{P}_\lambda\phi \neq 0$. Then,

$$\psi_\lambda = \frac{\hat{P}_\lambda\phi}{\left\|\hat{P}_\lambda\phi\right\|}$$

Thus,

$$\hat{P}_\lambda\psi_\lambda = \psi_\lambda$$

Now, to prove convergence, I write:

$$\hat{X}\psi_\lambda = \hat{X}\hat{P}_\lambda\psi_\lambda = \left(\int_0^{\mathbb{1}} \xi\,\mathrm{d}\hat{\mathbb{E}}\,(\xi)\right)\left(\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \mathrm{d}\hat{\mathbb{E}}\,(\eta)\right)\psi_\lambda$$

From equation (2), we get:

$$\hat{X}\psi_\lambda = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \xi\,\mathrm{d}\hat{\mathbb{E}}\,(\xi)\psi_\lambda$$

Thus,

$$\left\|\left(\hat{X} - \lambda\hat{\mathbb{1}}\right)\psi_\lambda\right\|^2 = \langle\psi_\lambda|\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (\xi-\lambda)^2\,\mathrm{d}\hat{\mathbb{E}}\,(\xi)\,|\psi_\lambda\rangle$$

In this, I have once again used equation (2). On the other hand,

$$\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (\xi-\lambda)^2\,\langle\psi_\lambda|\,\mathrm{d}\hat{\mathbb{E}}\,(\xi)\,|\psi_\lambda\rangle \leq \varepsilon^2\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \langle\psi_\lambda|\,\mathrm{d}\hat{\mathbb{E}}\,(\xi)\,|\psi_\lambda\rangle$$

The integral on the right side equals 1:

$$\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \langle\psi_\lambda|\,\mathrm{d}\hat{\mathbb{E}}\,(\xi)\,|\psi_\lambda\rangle = \langle\psi_\lambda|\left(\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \mathrm{d}\hat{\mathbb{E}}\,(\xi)\right)|\psi_\lambda\rangle = \langle\psi_\lambda|\,\hat{P}_\lambda\,|\psi_\lambda\rangle = 1$$

Finally, we obtain:

$$\boxed{\left\|\left(\hat{X} - \lambda\hat{\mathbb{1}}\right)\psi_\lambda\right\| \leq \varepsilon}$$

Similarly:

$$\left\|\left(f(\hat{X}) - f(\lambda)\hat{\mathbb{1}}\right)\psi_\lambda\right\| \leq \varepsilon|f'(\lambda)| + \mathcal{O}(\varepsilon^2)$$

3 My Question

We know that operators with continuous spectra do not have exact eigenvectors. The proof is simple:

$$x\psi(x) = \lambda\psi(x) \Rightarrow (x-\lambda)\psi(x) = 0$$

Therefore, $\psi(x) = 0$ for all x except $x = \lambda$. Since the Dirac delta distribution is not square-integrable, it's not a member of the Hilbert space. Hence, there doesn't exist such $\psi(x)$ satisfying the eigenvalue equation. On the other hand, the completeness axiom of Hilbert space asserts that if the norm of a sequence of vectors like ψ_n converges to a vector like ψ, then ψ itself is a member of Hilbert space.

Here, I proved that there exists a sequence of ψ_n's such that $\left\|\left(\hat{X}-\lambda\hat{\mathbb{1}}\right)\psi_\lambda\right\|$ converges to zero but there's not any vector in Hilbert space that satisfies $\left(\hat{X}-\lambda\right)\psi_\lambda = 0$. This seems to contradict the completeness of Hilbert space.

References

[1] Asher Peres, *Quantum Theory: Concepts and Methods*. Springer Dordrecht, 2002.

[2] Brian C. Hall, *Quantum Theory for Mathematicians*. Springer New York, NY, 2013.